Attractor invariants for local Calabi-Yau threefolds

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based on arXiv:2004.14466 with Guillaume Beaujard, Jan Manschot

and arXiv:2012.14358 with Sergey Mozgovoy

From math to physics and back

- Given a compact Calabi-Yau threefold X, one associates an infinite set of rational numbers n^0_β, called (genus zero) Gromov-Witten invariants, which count rational curves in homology class $\beta \in H_2(X)$. They are invariant under complex deformations of X, computable using mirror symmetry, and provide a deformation of the usual intersection product on $H_2(X)$.

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- In physics, genus zero GW invariants govern instanton corrections to the low-energy effective action in type IIA strings on $\mathbb{R}^{3,1} \times X$:

$$S[g, z, \dots] = \int_{\mathbb{R}^{3,1}} \sqrt{-\det g}\, \mathrm{d}^4 x \left[R(g) + G_{ab}(z) g^{\mu\nu} \partial_\mu z^a \partial_\nu z^b + \dots \right]$$

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- More precisely, corrections to the metric G_{ab} on Kähler moduli space $\mathcal{M}$ due to worldsheet instantons, i.e. Euclidean strings wrapped on rational curves. Higher genus GW invariants $n^{g>0}_\beta$ govern higher derivative corrections to $\mathcal{N} = 2$ supergravity.

From math to physics and back

- The genus zero Gopakumar-Vafa invariants defined by $N^0_\beta = \sum_{d|\beta} \frac{1}{d^3} n^0_{\beta/d}$ are conjecturally integer. In physics, they count BPS states originating from D2-branes wrapped on curves in homology class β. Higher genus GV invariants $N^{g>0}_\beta$ provide a refinement of that counting.

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- The Chern character $\gamma = \mathrm{ch}(E) \in H^{\mathrm{even}}(X)$ (or its Poincaré dual) is interpreted as the electromagnetic charge in $3+1$ dimensions. The shifted object $E[1]$ corresponds to the anti-particle with charge $-\gamma$.

From math to physics and back

- Physicists are interested in the BPS index $\Omega_z(\gamma)$ counting stable objects in $D(X)$ with fixed charge $\gamma \in \Gamma$ and moduli $z \in \mathcal{M}$. The mathematical counterpart is argued to be the generalized Donaldson-Thomas invariant, which is also of interest to many mathematicians ! *[Kontsevich 1994, Douglas 2001, Bridgeland 2005]*

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- While physicists can hardly compete with mathematicians in defining/computing $\Omega(\gamma, z)$ rigorously, they can use physics intuition to conjecture new properties of these invariants.
- For example, they expect that for large $|\gamma|$, stable objects in $D(X)$ correspond to BPS black hole solutions to $\mathcal{N} = 2$ supergravity. Moreover, $\log |\Omega_z(\gamma)| \sim \frac{1}{4}\mathcal{A}$ where $\mathcal{A}$ is the BH horizon area, measured in Planck units. Subleading corrections are in principle computable from higher-derivative corrections.

From math to physics and back

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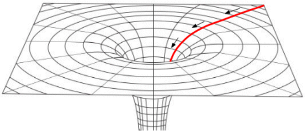
- Based on this picture, in earlier work with J. Manschot and A. Sen (2010), we derived the wall-crossing formula of Kontsevich and Soibelman by solving the supersymmetric (SUSY) quantum mechanics of n dyonic BPS black holes.
- By the Coulomb/Higgs correspondence, this turns out to be equivalent to computing DT invariants for the moduli space of stable representations of certain acyclic quivers. This problem was solved by M. Reineke (2003). Instead, we applied the Atiyah-Bott-Lefschetz fixed point theorem on the Coulomb branch to obtain a different (but equivalent) formula.

From math to physics and back

- Another physical insight is the idea of attractor mechanism: for single-centered black holes in $\mathcal{N}=2$ supergravity, the moduli have a non-trivial radial profile $z(r)$, interpolating from vacuum value z_∞ at $r=\infty$ to a value z_γ at the horizon, which depends only on the charge γ.



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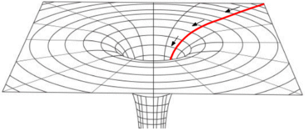

- The value z_γ at the horizon is determined by the condition

$$\forall\gamma' \quad \mathrm{Im}[Z_{\gamma'}\bar{Z}_\gamma](z_\gamma) = -\lambda_\gamma\langle\gamma',\gamma\rangle$$

where $Z_\gamma(z)$ is the central charge, $\langle -,-\rangle$ is the Dirac-Schwinger pairing and $\lambda_\gamma > 0$. In mathematical terms, $Z(z_\gamma)$ corresponds to the self-stability condition.

From math to physics and back

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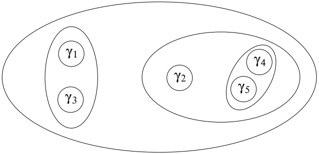

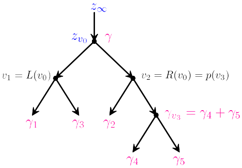

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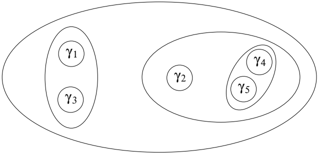

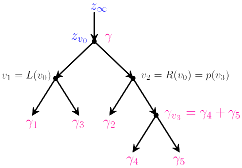

- This idea can be made precise, leading to 'attractor flow formulae' expressing $\Omega_z(\gamma)$ as a sum of products of attractor indices $\Omega_\star(\gamma_i)$.

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- In this talk, I will focus on the case where X is a crepant resolutions of toric CY3 singularity. The category $D(X)$ is equivalent to the category of representations of a certain quiver with potential $D(Q, W)$, associated to a brane tiling.

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- If true, this conjecture determines the entire set of DT invariants $\Omega_\theta(d)$ via the attractor flow tree formulae, which are now theorems due to Argüz, Bousseau and Mozgovoy.

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$$h_{d,\delta}(\tau) = \sum_{n=0}^{\infty} \Omega_{\theta=\langle d,-\rangle}(d+n\delta)\, q^{n+\Delta} , \qquad q = e^{2\pi i\tau}$$

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- A natural question is whether this can be understood from the action of some VOA on the cohomology of quiver moduli...

Definitions and notations [slide courtesy of S. Mozgovoy]

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- For any cycle $p = a_n \ldots a_1$ and for any arrow $a \in Q_1$, define the (cyclic) derivation $\frac{\partial p}{\partial a} = \sum_{i:a_i=a} a_{i+1} \ldots a_n a_1 \ldots a_{i-1}$.
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- Define the Jacobian algebra $J(Q, W) = \mathbb{C}Q/(\partial W/\partial a : a \in Q_1)$.
- Define a cut to be a subset $I \subset Q_1$ such that every term of W contains exactly one arrow from I. Setting $Q' = Q \backslash I$, let $J_I(Q, W) = \mathbb{C}Q'/(\partial W/\partial a : a \in I)$.

Brane tilings

- The quiver (Q, W) associated to a singular toric CY3 X is conveniently encoded in a brane tiling, i.e. a bipartite graph G embedded in a 2-dimensional (real) torus $\mathcal{T}$. Each vertex is black or white, edges connect only vertices of different colors.

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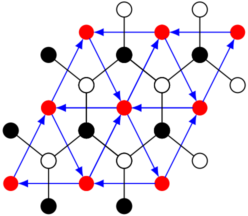

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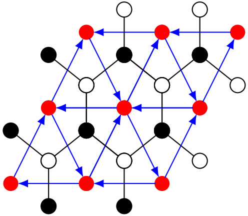

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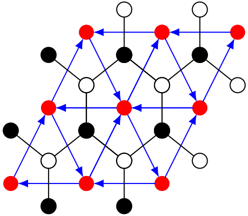

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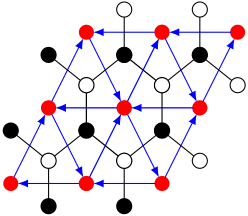

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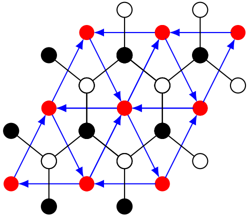
- For every $i \in Q_0$, as many arrows come in as come out.

Brane tilings II



1. Let $Q_2 = Q_2^+ \cup Q_2^-$ be the set of white and black vertices of G, or equivalently the set of faces of Q (connected components of $\mathcal{T}\backslash Q$),
2. For any face $F \in Q_2$, let w_F be the cycle obtained by going along the arrows of F (defined up to a cyclic shift).
3. The potential is $W = \sum_{F \in Q_2^+} w_F - \sum_{F \in Q_2^-} w_F$

- The quiver (Q, W) can be derived from a tilting sequence on X. Conversely, the toric diagram of X can be read off from zig-zag paths on the brane tiling. X arises as the moduli space of representations of (Q, W) with dimension vector $\delta = (1, 1, \dots)$.

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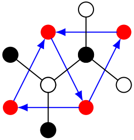

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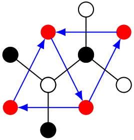

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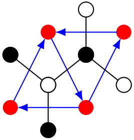

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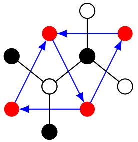

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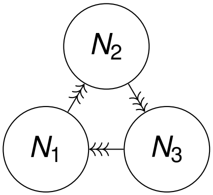
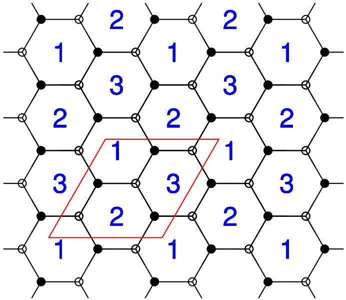
- For $I = \{z\}$, $J_I(Q, W) = \mathbb{C}[x, y]$ is the coordinate ring of $\mathbb{C}^2$.

Example 2: $\mathbb{C}^3/\mathbb{Z}_3 \sim K_{\mathbb{P}^2}$



$$W = \sum_{i,j,k} \epsilon_{ijk} \Phi^i_{12} \Phi^j_{23} \Phi^k_{31}$$



- For $I = \{\Phi^1_{31}, \Phi^2_{31}, \Phi^3_{31}\}$, the quiver $Q' = Q \backslash I$ with relations $\epsilon_{ijk} \sum_{j,k} \Phi^i_{12} \Phi^j_{23} = 0$ is the familiar Beilinson quiver describing coherent sheaves on $\mathbb{P}^2$.

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Stacky invariants

- Given a cut $I \subset Q_1$, we define the stacky DT invariants by

$$\mathcal{A}(x) = \sum_{d \in \mathbb{N}^{Q_0}} (-y)^{\chi_Q(d,d)+2\gamma_I(d)} \frac{P(R(J_I,d))}{P(G_d)} x^d$$

where $P(X) = \sum_n \dim H^n(X)(-y)^n$ for smooth projective X,

$$\chi_Q(d,e) = \sum_i d_i e_i - \sum_{a:i\to j} d_i e_j, \quad \gamma_I(d) = \sum_{(a:i\to j)\in I} d_i d_j$$

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- For any stability function Z and ray $\ell \subset \mathbb{C}$, define

$$\mathcal{A}_{Z,\ell}(x) = \sum_{d: Z(d) \in \ell} (-y)^{\chi_Q(d,d) + 2\gamma_I(d)} \frac{P(R_Z(J_I, d))}{P(G_d)} x^d .$$

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Wall-crossing

- Define the quantum torus

$$\mathbb{A} = \bigoplus_{d \in \mathbb{N}^{Q_0}} \mathbb{Q}(y) x^d \,, \quad x^d \circ x^{d'} = (-y)^{\langle d, d' \rangle} x^{d+d'}$$

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- Joyce has given a formula expressing $\bar\Omega_Z(d,y)$ in terms of $\bar\Omega_{Z'}(d',y)$ for all $d' \le d$.

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- For example, for $X = \mathbb{C}^3$

$$\mathcal{A}(x) = \mathrm{Exp}\left(\frac{-y^3 \sum_{n\geq 1} x^n}{y^{-1} - y}\right) \qquad \Rightarrow \Omega(n, y) = -y^3$$

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- For $d = n\delta$, $M_Z(J, d)$ is the Hilbert scheme of n points on X, and one has *[Behrend-Bryan-Szendroi (2009)]*

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- Similarly, for dimension vectors d associated to coherent sheaves supported on curves C which do not intersect the compact divisors, $\Omega_Z(d, y)$ is independent of Z, and coincides (in unrefined limit $y \to 1$) with the genus-zero Gopakumar-Vafa invariant N^0_β.

Attractor invariants

- Given a dimension vector $d \in \mathbb{N}^{Q_0}$, consider $\theta = \langle -, d \rangle : \mathbb{Z}^{Q_0} \to \mathbb{R}$ and let θ' be a generic perturbation. Theorem *[MP 2020, Gross Hackinng Keel Konstevich 2014]*: $\bar{\Omega}_{\theta'}(d, y)$ is independent of the perturbation.

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- Theorem: If Q is acyclic, then $\Omega_\star(d) = 1$ for $d = e_i$ and zero otherwise. More generally, if the support of d is not strongly connected, then $\Omega_\star(d) = 0$.
- Using the wall-crossing formulas, the DT invariants $\bar{\Omega}_Z(d, y)$ for any stability parameter Z can be recursively expressed in terms of attractor invariants.

Attractor flow tree formulae

- More directly, the attractor tree formula allows to express $\bar{\Omega}_\theta(\gamma, y)$ in terms of the attractor indices $\bar{\Omega}_\star(\alpha_i, y)$:

$$\bar{\Omega}_\theta(\gamma, y) = \sum_{\gamma=\sum \alpha_i} \frac{g_\theta(\{\alpha_i\}, y)}{|\mathrm{Aut}(\{\alpha_i\})|} \prod_i \bar{\Omega}_\star(\alpha_i, y)$$

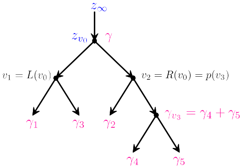
Manschot'10, Alexandrov BP '18; Argüz Bousseau '21

where

$$g_\theta(\{\alpha_i\}, y) = \sum_{T \in \mathcal{T}_\theta} \prod_{v \in V_T} (-1)^{\gamma_{LR}} \frac{y^{\gamma_{LR}} - y^{-\gamma_{LR}}}{y - 1/y}$$

Here T runs over all θ-stable flow trees ending on the leaves $\alpha_1, \ldots, \alpha_n$, v runs over all vertices and $\gamma_{LR} = \langle \gamma_{L(v)}, \gamma_{R(v)} \rangle$.

- To define stability, decorate each vertex v with a dimension vector γ_v and stability parameter θ_v, such that $\gamma_v = \alpha_i$ for the i-th leaf, $\theta_{v_0} = \theta$ for the root vertex, and for any v distinct from the root and the leaves, with parent $p(v)$ and descendants $L(v), R(v)$,



$$
\begin{aligned}
\gamma_v &= \gamma_{L(v)} + \gamma_{R(v)} \\
\theta_v &= \theta_{p(v)} + \frac{\langle \gamma_v, - \rangle}{\langle \gamma_{L(v)}, \gamma_v \rangle} \theta_{p(v)}(\gamma_{L(v)})
\end{aligned}
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Attractor flow and attractor indices

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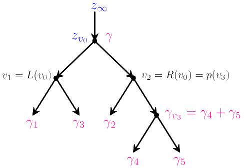


$$\begin{aligned} \gamma_v &= \gamma_{L(v)} + \gamma_{R(v)} \\ \theta_v &= \theta_{p(v)} + \frac{\langle \gamma_v, - \rangle}{\langle \gamma_{L(v)}, \gamma_v \rangle} \theta_{p(v)}(\gamma_{L(v)}) \end{aligned}$$

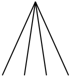
- The flow tree is θ-stable if $\langle \gamma_{L(v)}, \gamma_{R(v)} \rangle \times \theta_v(\gamma_{L(v)}) > 0$ for all v (after perturbing $\langle -, - \rangle$ or θ).

Attractor flow and attractor indices

- There is a different formula called flow tree formula which does not require any perturbations. It involves a sum over rooted plane trees with vertices of arbitrary valency, produces numerous cancellations and its physical interpretation is obscure. *[Alexandrov BP Manschot '19, Mozgovoy BP '20; Mozgovoy '20]*

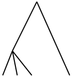
Attractor flow and attractor indices

- For example for Kronecker quiver K_m, $d = (1, 3)$,



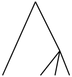
$\gamma_1\gamma_2\gamma_2\gamma_2$

$a : -\frac{1}{4}(-y)^{3m}$



$\gamma_1\gamma_2\gamma_2\gamma_2$

$b : \frac{1}{4}(-y)^{3m}$

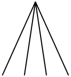


$\gamma_1\gamma_2\gamma_2\gamma_2$

$c : \frac{1}{12}(-y)^{3m}$

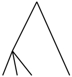
$\gamma_1\gamma_2\gamma_2\gamma_2$

$d : -\frac{1}{4}(-y)^{3m}$



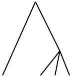
$\gamma_2\gamma_1\gamma_2\gamma_2$

$e : \frac{1}{4}(-y)^{m}$



$\gamma_2\gamma_1\gamma_2\gamma_2$

$f : -\frac{1}{4}(-y)^{m}$



$\gamma_2\gamma_1\gamma_2\gamma_2$

$g : -\frac{1}{4}(-y)^{m}$

$\gamma_2\gamma_1\gamma_2\gamma_2$

$h : \frac{1}{4}(-y)^{m}$

$\gamma_2\gamma_1\gamma_2\gamma_2$

$i : \frac{1}{4}(-y)^{m}$

$\gamma_2\gamma_1\gamma_2\gamma_2$

$j : \frac{1}{4}(-y)^{m}$

$$+(y \to \tfrac{1}{y}) = -\frac{1}{6}\left((-y)^m - (-y)^{-m}\right)^3$$

Attractor indices from stacky invariants

- One way to determine the attractor invariants $\Omega_\star(d, y)$ is to compute the stacky invariants for trivial stability $\mathcal{A}(d, y)$ and apply the wall-crossing formula.

Attractor indices from stacky invariants

- One way to determine the attractor invariants $\Omega_\star(d, y)$ is to compute the stacky invariants for trivial stability $\mathcal{A}(d, y)$ and apply the wall-crossing formula.
- For quivers associated to brane tilings, $\mathcal{A}(d, y)$ can be computed using double dimensional reduction. Let I and I' be two disjoint cuts, and let $Q' = Q \backslash I$, $Q'' = Q \backslash (I \cup I')$. There is a forgetful map $\pi : R(J_I, d) \to R(Q'', d)$ with linear fibers. Thus $\mathcal{A}(d, y)$ can be deduced from the set of indecomposable representations $\mathcal{R}$ of Q'':

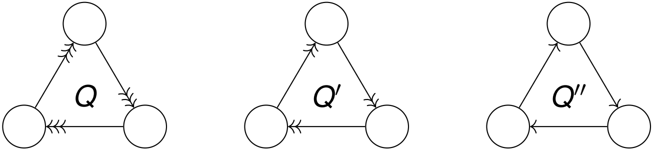
$$\mathcal{A}(x) = \sum_{m:\mathcal{R}\to\mathbb{N}} \frac{(-y)^{-\sum_{M,N\in\mathcal{R}} m_M m_N \sigma(M,N)}}{\prod_{M\in\mathcal{R}}[GL(m_M)]} x^{\sum_{m\in\mathcal{R}} m_M \dim M}$$

$$\sigma(M, N) = 2 \dim \mathrm{Hom}(M, N) - \phi(M, N) - \chi_Q(M, N) - 2\gamma_I(M, N)$$

where $\phi(M, N)$ is the quad. form such that $\phi(M, M) = \dim \pi^{-1}(M)$.

Attractor indices from stacky invariants

- Consider $K_{\mathbb{P}^2}$:



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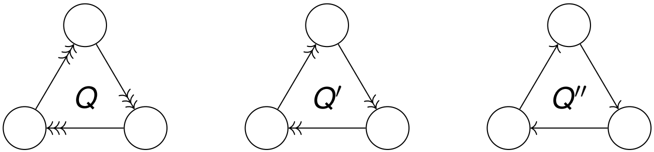


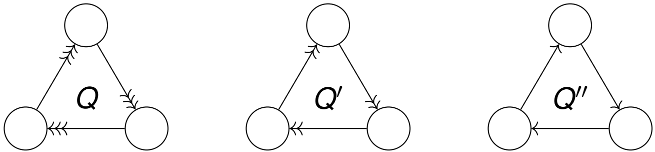
- This suggests the conjecture

$$\Omega_*(e_i, y) = 1, \qquad \Omega_*(n\delta, y) = -y^3 - y - y^{-1}$$

and all other invariants vanish.

Attractor indices from stacky invariants

- Consider $K_{\mathbb{P}^2}$:



- This suggests the conjecture

$$\Omega_*(e_i, y) = 1, \qquad \Omega_*(n\delta, y) = -y^3 - y - y^{-1}$$

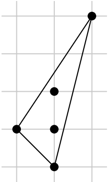
and all other invariants vanish.
- The same type of computation for $K_{\mathbb{F}_0}, K_{\mathbb{F}_1}, K_{dP_2}, \mathbb{C}^3/\mathbb{Z}_5$, ...suggests a similar conjecture for isolated toric CY3 singularities, with

$$\Omega_\star(n\delta, y) = (-y)^3 P(X) = -y^3 - (i + b - 3)y - iy^{-1}$$

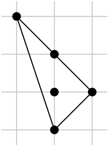
where i (resp. b) are the number of internal (resp. boundary) lattice points on the toric diagram.

Attractor indices from stacky invariants

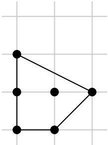
- For non-isolated toric singularities, such that the boundary of the toric diagram contains lattice points beyond the corners, we find $\Omega_\star(d + n\delta, y) = -y$ for some d in the kernel of $\langle -, - \rangle$. See *[Descombes (2021)]* for a precise conjecture covering all brane tilings.



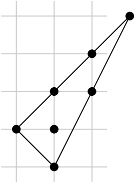
$\mathbb{C}^3/\mathbb{Z}_5$



$\mathbb{F}_2$



PdP_2



$\mathbb{C}^3/\mathbb{Z}_6(1, 2, 3)$

Attractor invariants from NCDT invariants

- One can also compute $\Omega_\star(d, y)$ from framed DT invariants in the non-commutative chamber, counting D4-D2-D0 branes bound to an infinitely heavy D6-brane

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- For any framing vector $\mathrm{f} \in \mathbb{N}^{Q_0}$, let Q^{f} be the quiver obtained from Q by adding a new vertex ∞ and f_i arrows $\infty \to i$, for $i \in Q_0$.
- Let $J^{\mathrm{f}} = J(Q^{\mathrm{f}}, W)$, $d^{\mathrm{f}} = (d,1)$ and $R^{\mathrm{f}}(J,d) = R(J^{\mathrm{f}}, d^{\mathrm{f}})$. Let $R^{\mathrm{f,NC}}(J,d) \subset R^{\mathrm{f}}(J,d)$ to be the subspace of cyclic representations M (i.e. satisfying $N \subset M, N_\infty \neq 0 \Rightarrow N = M$).

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- We define the generating function of unrefined NCDT invariants

$$Z_{\mathrm{NC}}(x) = \sum_{d \in \mathbb{N}^{Q_0}} (-1)^{\chi_Q(d,d) - \mathrm{f}\cdot d} e\left(R^{\mathrm{f,NC}}(J, d)/G_d\right) x^d$$

Attractor invariants from NCDT invariants

- NCDT invariants are related to (unframed, unrefined) DT invariants by wall-crossing. The formula is simplest for symmetric quivers,

$$Z_{\mathrm{f},\mathrm{NC}}(x) = \bar{S}_{\mathrm{f}}\,\mathrm{Exp}\left(-\sum_{d\in\mathbb{N}^{Q_0}} (\mathrm{f}\cdot d)\Omega(d,1)x^d \right) , \quad S_{\mathrm{f}}(x^d) = (-1)^{\mathrm{f}\cdot d} x^d$$

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- NCDT invariants can be computed using toric localization, which amounts to counting molten crystals.

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- Theorem *[Mozgovoy Reineke (2008)]*:

$$Z_{e_i,\mathrm{NC}}(x) = \sum_{\mathcal{I}\subset\Delta_i, d=\dim\mathcal{I}} (-1)^{\chi_Q(d,d)+d_i} x^d$$

where $\dim\mathcal{I} = \sum_{u\in\mathcal{I}} e_{t(u)} \in \mathbb{Z}^{Q_0}$.

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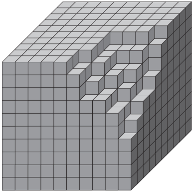
- Z_f for other framing vectors $f \in \mathbb{N}^{Q_0}$ follows from $Z_{f+f'} = Z_f Z_{f'}$.

Attractor invariants from molten ctrystals



Example: For $X = \mathbb{C}^3$, with Jacobian algebra $J(Q,W) = \mathbb{C}[x,y,z]$, one can identify the poset Δ_1 with $\mathbb{N}^3$, and ideals with plane partitions.

- The generating function of NCDT invariants is *[MacMahon 1916]*

$$Z_1(-x) = \prod_{k=1}^{\infty} (1-x^k)^{-k} = 1+x+3x^2+6x^3+13x^4+24x^5+48x^6+\dots$$

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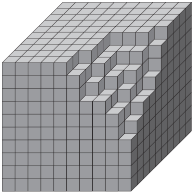

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- Using this approach we have confirmed the Attractor Conjecture for all brane tilings in the unrefined limit. The computation of refined NCDT invariants by toric localization is much harder, but confirms the conjecture. *[Descombes (2021)]*

Attractor invariants from Vafa-Witten invariants

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- Remarkably, for the canonical polarization $J \propto c_1(S)$, $\theta_i \propto \kappa_{ij} d_j$ corresponds to the anti-attractor stability condition !

Attractor invariants from Vafa-Witten invariants

- The VW invariants $c_J^S([N, \mu, n])$ for any rational surface S, polarization J, rank N, first Chern class μ and second Chern class n can be computed by combining blow up and wall-crossing formulae *[Yoshioka (1996), Goettsche (1999), Manschot (2011-14)]*.

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- Comparing the first few coefficients for $J = c_1(S)$ with the anti-attractor indices for the quiver (Q,W) computed by assuming the Attractor conjecture, we find perfect agreement for toric Fano surfaces $\mathbb{P}^2, \mathbb{F}_0, dP_{1\le n\le 3}$, for a variety of brane tilings.

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- Agreement persists for non-toric Fano surfaces $dP_{4\le n\le 8}$.

Anti-attractor invariants and modularity

- For fixed rank N and first Chern class μ, the generating series $h^S_{N,\mu,J}(\tau, z)$ is expected to be quasi-invariant under $SL(2,\mathbb{Z})$ transformations: $\tau \to \frac{a\tau+b}{c\tau+d}, z \to \frac{z}{c\tau+d}$. This follows from Montonen-Olive S-duality of $\mathcal{N} = 4$ super Yang-Mills theory.

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- This anomalous transformation properties can be repaired at the cost of adding non-holomorphic corrections, determined in terms of mock Jacobi forms of lower depth.

Anti-attractor invariants and modularity

- Translating into the language of quivers, this suggests that the generating function of anti-attractor DT invariants

$$h_{d,\delta}(\tau) = \sum_{n=0}^{\infty} \Omega_{\langle d,-\rangle}(d+n\delta)\, q^{n+\Delta} \ , \qquad q = e^{2\pi i \tau}$$

  for δ a primitive vector such that $\kappa \cdot \delta = 0$ and a suitable $\Delta \in \mathbb{Q}$, should be a vector-valued mock modular form of depth $\chi_Q(d,\delta) - 1$. For $\chi_Q(d,\delta) = 1$, it should be truly modular.

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- Thank you for your attention.