Attractor indices, brane tilings and crystals

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based on arXiv:2004.14466 with Guillaume Beaujard, Jan Manschot
and arXiv:2012.14358 with Sergey Mozgovoy

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- Part of the reason is that $\Omega(\gamma, z)$ depends on the moduli z in an intricate way, due to wall-crossing phenomena associated to BPS bound states with any number of constituents. The moduli space itself receives quantum corrections, unlike in $\mathcal{N} \geq 4$.

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- Alternatively, in type IIB/$X^\vee$, the BPS indices $\Omega(\gamma, z)$ counting D3-brane bound states with charge $\gamma \in H^3(X^\vee)$ are the DT invariants of the Fukaya category of lagrangian submanifolds of $X^\vee$, and are invariant under Kähler deformations of $X^\vee$. Mirror symmetry equates the two sets of invariants.

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- For D6-D2-D0 bound states for single unit of D6-brane charge at large volume, $\Omega(\gamma, z)$ are the standard Donaldson-Thomas invariants, determined by the same GV invariants.

  Thomas' 99; Maulik Nekrasov Okounkov Panharipande '04

- D4-D2-D0 black holes can be realized by wrapping an M5 on a compact 4-cycle $P \subset X$, hence are described by a 2D superconformal field theory. The generating series of BPS indices is expected to be modular under $SL(2,\mathbb{Z})$. The central charge of the SCFT predicts the correct entropy at large charge, but exact indices are known only in a handful of cases.

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- Alternatively, by reducing along T^2, D4-D2-D0 branes on a rigid divisor P are described by Vafa-Witten theory on P. Unless P is irreducible, the generating series of VW invariants is expected to be a (vector-valued) mock modular form, with a precise modular anomaly.

Minahan Nemeschansky Vafa Warner'98

Alexandrov Banerjee Manschot BP'16-19; Dabholkar Putrov Witten '20

Toric CY3, quivers and brane tilings

- In this talk, I will consider BPS states in type IIA string theory compactified on a non-compact toric CY threefold. In that case, the category of branes $\mathcal{D}(X)$ is isomorphic to the category of representations of a certain quiver with superpotential (Q, W).

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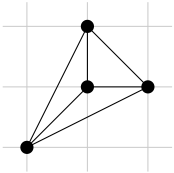
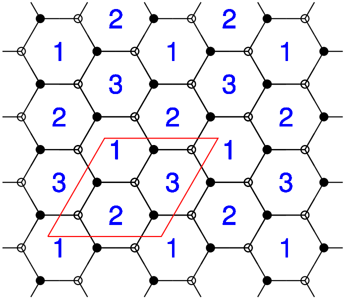
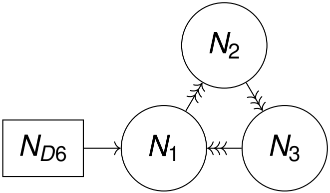
- The quiver Q and superpotential W are conveniently summarized by a brane tiling, or equivalently a periodic quiver.

Franco Hanany Kenneway Vegh Wecht'05

Example: $\mathbb{C}^3/\mathbb{Z}_3 \sim K_{\mathbb{P}^2}$







$$W = \sum_{i,j,k} \epsilon_{ijk} \Phi^i_{12} \Phi^j_{23} \Phi^k_{31}$$

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$$\zeta^\star_k(d) := -\sum_\ell \kappa_{k\ell} N_\ell \qquad \Omega_\star(d) := \Omega(d, \zeta^\star(d))$$

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- More generally, $\Omega_\star(d) = 0$ unless d is supported on a strongly connected subquiver. *[Manschot BP Sen '13, unpublished]*

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- More generally, for toric CY3 singularities, we claim that $\Omega_*(d) = 0$ unless $d_a = \delta_{a,\ell}$ or d lies in (a subspace of) the kernel of the Dirac pairing (i.e. $\langle d, d' \rangle = 0$ for all d').

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- For refined DT invariants, the same conjecture holds with $\Omega_\star(n\delta, y) = (-y)^{-3}[X] = -b_0 y^3 - b_2 y - b_4/y - b_6/y^3$.

Attractor flow tree formula

- If correct, this conjecture gives access to the BPS indices $\Omega(\gamma, z)$ for any γ, z by applying wall-crossing formula, or more efficiently by using the attractor flow tree formula(e)

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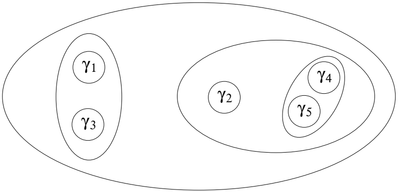
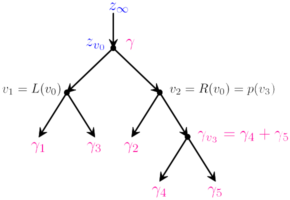
- In short, the full BPS spectrum arises from bound states of these attractor BPS states, with a hierarchical structure labelled by attractor flow trees:





Attractor flow tree formulae

- The attractor tree formula allows to express $\bar{\Omega}(\gamma, y, \zeta)$ in terms of the attractor indices $\bar{\Omega}_\star(\alpha_i, y) := \bar{\Omega}(\alpha_i, y, \zeta^*(\alpha_i))$

$$\bar{\Omega}(\gamma, y, \zeta) = \sum_{\gamma = \sum \alpha_i} \frac{g_{\rm tr}(\{\alpha_i\}, y, \zeta)}{|{\rm Aut}(\{\alpha_i\})|} \prod_i \bar{\Omega}_*(\alpha_i, y)$$

Manschot'10, Alexandrov BP '18

where

$$g_{\rm tr}(\{\alpha_i\}, y, \zeta) = \sum_T \prod_{v \in V_T} (-1)^{\gamma_{LR}} \frac{y^{\gamma_{LR}} - y^{-\gamma_{LR}}}{y - 1/y}$$

Here T runs over all possible stable flow trees T ending on the leaves $\alpha_1, \ldots, \alpha_n$, v runs over all vertices and $\gamma_{LR} = \langle \gamma_{L(v)}, \gamma_{R(v)} \rangle$.

Support for the Attractor Conjecture

The conjecture is supported by computing

- for $X = K_S$, the BPS indices for D4-D2-D0 branes wrapped on S, and comparing with known results for Vafa-Witten invariants on S.

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- for any brane tiling, the framed BPS indices for D6-D4-D2-D0 branes in the non-commutative chamber, and comparing with the combinatorics of molten crystals

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- Other arguments, including computations of DT invariants for trivial stability condition, invariance under mutations, and exponential (spectral) networks.

  Bonelli del Monte Tanzini '20, Banerjee Longhi Romo '20

General comments

- The simplicity of attractor invariants is a special feature of quivers attached to CY3 singularities. It depends on the special form of W. Toricity does not seem essential, since the Attractor Conjecture appears to hold also for K_{dP_n} with $4 \leq n \leq 8$.

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- The fact that $\Omega_\star(\gamma) = \mathcal{O}(1)$ is perhaps disappointing but consistent with gravity being decoupled. Instead, it seems to be a general property of the BPS spectrum of 4D/5D gauge theories constructed by geometric engineering. *[Closset del Zotto '19]*

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- How about modularity ? compact CY threefolds ?

Outline

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Quiver quantum mechanics

- Consider a SUSY quantum mechanics in $0+1$ dimensions, obtained by reducing $\mathcal{N}=1$ gauge theory in $3+1$ dimension, with matter content encoded in a quiver: each node $\ell = 1...K$ represents a $U(d_\ell)$ vector multiplet, each arrow from k to ℓ represents a chiral multiplet $\Phi^{\alpha}_{k,\ell}$ in $(d_\ell, \bar{d}_k)$ representation of $U(d_\ell) \times U(d_k)$. *[Denef '02]*

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- The ranks $\{d_\ell\}$ are encoded in a dimension vector $\gamma = \sum d_\ell \gamma_\ell$ in a lattice Γ, endowed with an antisymmetric Dirac-Schwinger pairing $\langle \gamma, \gamma' \rangle = \sum \kappa_{k\ell} d_k d'_\ell$ where $\kappa_{k\ell}$ is the skew-adjacency matrix (the number of arrows from node k to node ℓ counted with sign).

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- In addition, one must specify Fayet-Iliopoulos terms $\zeta_\ell \in \mathbb{R}$ and (in presence of closed oriented loops) a superpotential $W(\Phi)$.

Quiver quantum mechanics

- On the Higgs branch, the moduli space of classical SUSY vacua $\mathcal{M}_H(\gamma,\zeta)$ is the solutions of the F-term and D-term equations modulo set gauge equivalence,

$$\forall \ell : \sum_{\gamma_{\ell k}>0} \Phi^*_{\ell k}\, T^a\, \Phi_{\ell k} - \sum_{\gamma_{k\ell}>0} \Phi^*_{k\ell}\, T^a\, \Phi_{k\ell} = \zeta_\ell\, \mathrm{Tr}(T^a)$$

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- 'stable' means that $\mu(\gamma') < \mu(\gamma)$ for any proper subrepresentation with dimension vector $\gamma' < \gamma$, where $\mu(\gamma') = (\sum_\ell \zeta_\ell d'_\ell)/\sum d'_\ell$ is the slope. *[King'94]*

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- BPS states correspond to Dolbeault cohomology classes of degree (p,q) on in $\mathcal{M}_H(\gamma,\zeta)$, counted by the Hodge polynomial

$$\Omega(\gamma,y,t,\zeta) = \sum_{p,q=0}^{2d} h_{p,q}(\mathcal{M}_H(\gamma,\zeta))\,(-y)^{p+q-d}\,t^{p-q}$$

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- $\Omega(\gamma,y)$ also counts BPS states on the Coulomb branch, but that interpretation is subtle due to scaling solutions.

Primitive wall-crossing

- The DT invariants $\Omega(\gamma, y, \zeta)$ for $\gamma \in \mathrm{Span}(\gamma_1, \gamma_2)$ jump on walls where $\mu(\gamma_1) = \mu(\gamma_2)$. For primitive dimension vectors $\gamma_{1,2}$ with Dirac-Schwinger pairing $\gamma_{12} = \langle \gamma_1, \gamma_2 \rangle$,

$$\Delta\Omega(\gamma_1 + \gamma_2, y) = (-1)^{\gamma_{12}} \frac{y^{\gamma_{12}} - y^{-\gamma_{12}}}{y - 1/y} \Omega(\gamma_1, y)\Omega(\gamma_2, y)$$

Physically, a two-centered bound state with spin degeneracy $2j + 1 = |\gamma_{12}|$ appears/disappears. *[Denef Moore '07]*

Primitive wall-crossing

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  Physically, a two-centered bound state with spin degeneracy $2j + 1 = |\gamma_{12}|$ appears/disappears. *[Denef Moore '07]*

- For more general charges, it is useful to introduce the rational invariants

$$\bar{\Omega}(\gamma, y) = \sum_{m|\gamma} \frac{1}{m} \frac{y - 1/y}{y^m - 1/y^m} \Omega(\gamma/m, y^m)$$

General wall-crossing

- The discontinuity across the hyperplane where $\mu(\gamma_1) = \mu(\gamma_2)$ is then given by a universal wall-crossing formula.

Konsevitch Soibelman'08, Joyce Song'08

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- On physical grounds, we expect and get

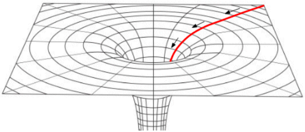
$$\bar{\Omega}(\gamma, y, \zeta_+) = \sum_{\gamma = \sum \alpha_i} \frac{g_{\rm WC}(\{\alpha_i\}, y)}{|\mathrm{Aut}(\{\alpha_i\})|} \prod_i \bar{\Omega}(\alpha_i, y, \zeta_-)$$

where $|\mathrm{Aut}(\{\alpha_i\})|$ is a Boltzmann symmetry factor, and $g_{\rm WC}(\{\alpha_i\}, y)$ is the index for Abelian quiver quantum mechanics with one node v_i for each α_i, and $\langle \alpha_i, \alpha_j \rangle$ arrows from v_i to v_j. This is computable using localisation on the Coulomb branch, or using Reineke's formula on the Higgs branch.

Reineke '02; Manschot BP Sen '10

Attractor flow and attractor indices

- For spherically symmetric black holes in $\mathcal{N}=2$ supergravity, the moduli flow radially from z_∞ to a value z_γ determined by the attractor mechanism: for $\gamma=(p^\lambda,q_\lambda)$,



$$\begin{array}{rcl} \mathrm{Im}[e^{-\mathrm{i}\alpha} X^\Lambda(z_\gamma)] & = & p^\Lambda \\ \mathrm{Im}[e^{-\mathrm{i}\alpha} F_\Lambda(z_\gamma)] & = & q_\Lambda \\ \Rightarrow \forall\gamma'\ \mathrm{Im}[e^{-\mathrm{i}\alpha} Z_{\gamma'}(z_\gamma)] & = & -\langle\gamma',\gamma\rangle \end{array}$$

Ferrara Kallosh Strominger'95

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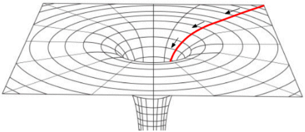

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Ferrara Kallosh Strominger'95

- Similarly, in quiver quantum mechanics there is a choice of stability parameters where (non-scaling) bound states are ruled out,

$$\zeta_k^\star(\gamma) = -\sum_\ell \gamma_{k\ell} d^\ell = -\langle\gamma_k,\gamma\rangle$$

known as attractor point or self-stability *[Manschot BP Sen '13, unpublished]*

Attractor flow tree formulae

- The attractor tree formula allows to express $\bar{\Omega}(\gamma, y, \zeta)$ in terms of the attractor indices $\bar{\Omega}_\star(\alpha_i, y) := \bar{\Omega}(\alpha_i, y, \zeta^*(\alpha_i))$

$$\bar{\Omega}(\gamma, y, \zeta) = \sum_{\gamma=\sum \alpha_i} \frac{g_{\rm tr}(\{\alpha_i\}, y, \zeta)}{|{\rm Aut}(\{\alpha_i\})|} \prod_i \bar{\Omega}_*(\alpha_i, y)$$

Manschot'10, Alexandrov BP '18

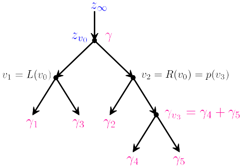
where

$$g_{\rm tr}(\{\alpha_i\}, y, \zeta) = \sum_T \prod_{v \in V_T} (-1)^{\gamma_{LR}} \frac{y^{\gamma_{LR}} - y^{-\gamma_{LR}}}{y - 1/y}$$

Here T runs over all possible stable flow trees T ending on the leaves $\alpha_1, \ldots, \alpha_n$, v runs over all vertices and $\gamma_{LR} = \langle \gamma_{L(v)}, \gamma_{R(v)} \rangle$.

Attractor flow and attractor indices

- To define stability, decorate each vertex v with a dimension vector γ_v and stability parameter ζ_v, such that $\gamma_v = \alpha_i$ for the i-th leaf, $\zeta_{v_0} = \zeta$ for the root vertex, and for any v distinct from the root and the leaves, with parent $p(v)$ and descendants $L(v)$, $R(v)$,



$$
\begin{aligned}
\gamma_v &= \gamma_{L(v)} + \gamma_{R(v)} \\
\zeta_v &= \zeta_{p(v)} + \frac{\langle \gamma_v, - \rangle}{\langle \gamma_{L(v)}, \gamma_v \rangle} \zeta_{p(v)}(\gamma_{L(v)})
\end{aligned}
$$

The flow tree is stable if $\langle \gamma_{L(v)}, \gamma_{R(v)} \rangle \times \zeta_v(\gamma_{L(v)}) > 0$ for all v (after perturbing $\langle -, \rangle$).

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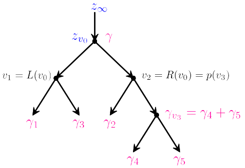

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- The attractor tree formula and its variant called flow tree formula are now mathematical theorems. *[Mozgovoy '20, Argüz Bousseau '21]*

Outline

Toric CY3 and brane tilings

- Toric CY3 are non-compact CY three-folds which admit an action of $(\mathbb{C}^\times)^3$ having a dense orbit. The category of coherent sheaves $\mathcal{D}(X)$ is isomorphic to the category of representations $\mathcal{D}(Q, W)$ of a quiver with superpotential.

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- The quiver (Q, W) are conveniently summarized by a brane tiling, i.e. a bipartite graph embedded in a two-torus. Tiles correspond to gauge groups, edges to chiral fields, and black/white vertices to monomials in the superpotential. The dual graph is a periodic quiver $\tilde{Q}$ covering Q.

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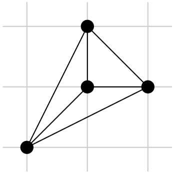
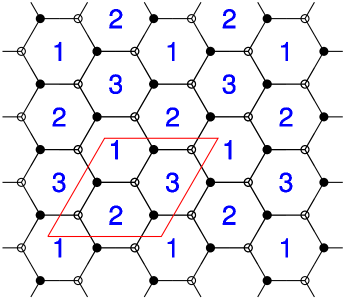
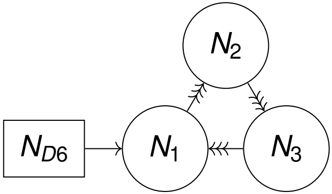
- The same toric CY3 may be described by different tilings/quivers, related by Seiberg duality.

Example: $\mathbb{C}^3/\mathbb{Z}_3 \sim K_{\mathbb{P}^2}$







$$W = \epsilon_{ijk} \Phi^i_{12} \Phi^j_{23} \Phi^k_{31}$$

Outline

Quivers from exceptional collections

- For local surfaces $X = K_S$, a basis of branes on $\mathcal{D}(X)$ (aka tilting sequence) can be constructed from an exceptional collection on S, i.e. an ordered sequence of (virtual) sheaves $(E_1, \dots, E_r)$ s.t.

$$\begin{aligned} \mathrm{Hom}(E_k, E_k) &= \mathbb{C}\,, \quad \mathrm{Ext}^m_S(E_k, E_k) \quad \forall m > 0 \\ \mathrm{Ext}^m_S(E_k, E_\ell) &= 0 \quad \forall (m \geq 0,\ 1 \leq \ell < k \leq r) \end{aligned}$$

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- There are two types of arrows $k \to \ell$: forward arrows from $\mathrm{Ext}^1(E_k, E_\ell)$ with $k < \ell$ and backward arrows from $\mathrm{Ext}^2(E_\ell, E_k)$ with $k > \ell$. The net number $\kappa_{k\ell}$ is computable from the Euler form

$$\chi(E, E') = \sum_{m \geq 0} (-1)^m \dim \mathrm{Ext}^m_S(E, E') = \int_S \mathrm{ch}(E^*)\,\mathrm{ch}(E')\,\mathrm{Td}(S)$$

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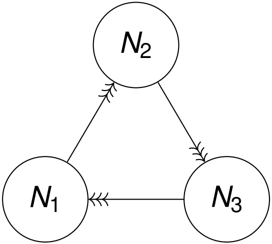
- The dimension vector d and FI parameters ζ can be related to the Chern vector γ and moduli z using $\gamma = \sum N_\ell \gamma_\ell$, $\zeta_\ell = \mathrm{Im}[Z_\gamma \overline{Z_{\gamma_\ell}}]$.

Sheaves on $\mathbb{P}^2$



$$
\begin{array}{ll}
E_1 = \mathcal{O} & \gamma_1 = [1,0,0] \\
E_2 = \Omega(1)[1] & \gamma_2 = [-2,1,\tfrac{1}{2}] \\
E_3 = \mathcal{O}(-1)[2] & \gamma_3 = [1,-1,\tfrac{1}{2}]
\end{array}
$$

$$
\chi(E_k, E_\ell) = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & -3 & 1 \end{pmatrix}
$$

[Le Potier'94]

Dimension vector: ($\propto (1,1,1)$ for D0-branes)

$$
(N_1, N_2, N_3) = -\left(\tfrac{3}{2}c_1 + \mathrm{ch}_2 + \mathrm{rk}, \tfrac{1}{2}c_1 + \mathrm{ch}_2, -\tfrac{1}{2}c_1 + \mathrm{ch}_2\right)
$$

For canonical polarization $J = \rho c_1(S)$ with $\rho \gg 1$,

$$
\zeta = 3\rho\,(N_2 - N_3, N_3 - N_1, N_1 - N_2) + \left(-\frac{N_2+N_3}{2}, \frac{N_1+3N_3}{2}, \frac{N_1-3N_2}{2}\right)
$$

Canonical vs. attractor chamber

- For any $X = K_S$, the canonical chamber $J = \rho\, c_1(S)$ in the large volume limit translates into the anti-attractor chamber,

$$\zeta_k = \rho \sum_{\ell} \kappa_{k\ell}\, N_\ell + \mathcal{O}(1)$$

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- The dimension of the moduli space of stable sheaves coincides with the dimension of the moduli space of stable representations,

$$d_{\mathbb{C}} = \sum_{a\notin I} N_k N_\ell - \sum_{a\in I} N_k N_\ell - \sum N_\ell^2 + 1 = 1 - \chi(E, E)$$

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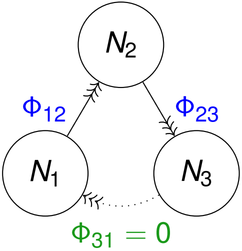
- In contrast, in the attractor chamber $\zeta_k = -\rho \sum_\ell \kappa_{k\ell} N_\ell$, the expected dimension is always negative, unless the dimension vector is one of the basis vectors γ_ℓ, or lies in the kernel of $\langle -, - \rangle$.



$$
\begin{aligned}
N_1 &= -\left(\tfrac{3}{2}c_1 + \mathrm{ch}_2 + \mathrm{rk}\right)\\
N_2 &= -\left(\tfrac{1}{2}c_1 + \mathrm{ch}_2\right)\\
N_3 &= -\left(-\tfrac{1}{2}c_1 + \mathrm{ch}_2\right)
\end{aligned}
$$

• In canonical (anti-attractor chamber), the expected dimension is positive at large instanton number $c_2 \sim -\mathrm{ch}_2$,

$$
\begin{aligned}
d_{\mathbb{C}} =& 3(N_1 N_2 + N_2 N_3 - N_3 N_1) - N_1^2 - N_2^2 - N_3^2 + 1\\
=& c_1^2 - 2\,\mathrm{rk}\,\mathrm{ch}_2 - \mathrm{rk}^2 + 1
\end{aligned}
$$

This requires $\zeta_1 \geq 0, \zeta_3 \leq 0$ hence $-\mathrm{rk} \leq c_1 \leq 0$.

Attractor invariants for $K_{\mathbb{P}^2}$

- In attractor chamber $\zeta^\star = 3\,(N_2 - N_3, N_3 - N_1, N_1 - N_2)$, the expected dimension is almost always negative:

$$d_{\mathbb{C}}^\star = 1 - \mathcal{Q}(\gamma) + \begin{cases} \frac23 N_3 \zeta_3^\star - \frac23 N_1 \zeta_1^\star & \zeta_1^\star \geq 0, \zeta_3^\star \leq 0 \quad (\Phi_{31} = 0) \\ \frac23 N_1 \zeta_1^\star - \frac23 N_2 \zeta_2^\star & \zeta_2^\star \geq 0, \zeta_1^\star \leq 0 \quad (\Phi_{12} = 0) \\ \frac23 N_2 \zeta_2^\star - \frac23 N_3 \zeta_3^\star & \zeta_3^\star \geq 0, \zeta_2^\star \leq 0 \quad (\Phi_{23} = 0) \end{cases}$$

$$\mathcal{Q}(\gamma) = \frac12 (N_1 - N_2)^2 + \frac12 (N_2 - N_3)^2 + \frac12 (N_3 - N_1)^2$$

hence $d_{\mathbb{C}}^* < 0$ unless $\gamma \in \{(1,0,0), (0,1,0), (0,0,1), (n,n,n)\}$.

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- We conjecture that $\Omega_\star(\gamma)=0$ except in those cases. We set $\Omega_\star(1,0,0)=\Omega_\star(0,1,0)=\Omega_\star(0,0,1)=1$, The index $\Omega_\star(n,n,n)$, corresponding to n D0-branes will be specified later.

VW invariants on $\mathbb{P}^2$

- Using the flow tree formula, and assuming the conjecture, we find that the index in canonical chamber agrees with VW invariants on $\mathbb{P}^2$ previously computed using blow-up/wall-crossing formulae !

Goettsche'90, Klyachko'91, Yoshioka'94, Manschot'11-14

$[N; c_1; c_2]$	(N_1, N_2, N_3)	$\Omega(\gamma, -\zeta^\star(\gamma))$
$[1;0;2]$	$(1,2,2)$	$y^4 + 2y^2 + 3 + \dots$
$[1;0;3]$	$(2,3,3)$	$y^6 + 2y^4 + 5y^2 + 6 + \dots$
$[2;0;3]$	$(1,3,3)$	$-y^9 - 2y^7 - 4y^5 - 6y^3 - 6y - \dots$
$[2;-1;2]$	$(1,2,1)$	$y^4 + 2y^2 + 3 + \dots$
$[2;-1;3]$	$(2,3,2)$	$y^8 + 2y^6 + 6y^4 + 9y^2 + 12 + \dots$
$[3;-1;3]$	$(1,3,2)$	$y^8 + 2y^6 + 5y^4 + 8y^2 + 10 + \dots$
$[4;-2;4]$	$(1,3,1)$	$y^5 + y^3 + y + \dots$

Attractor invariants for Fano surfaces

- We conjecture that the vanishing of attractor invariants holds for any CY threefold $X = K_S$ where S is a Fano surface. This includes the toric cases $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $S = dP_{k\leq 3}$, but also the non-toric del Pezzo surfaces $dP_{4\leq k\leq 8}$.

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- For those cases, we have computed VW invariants using the flow tree formula, under the assumption that $\Omega_\star(\gamma, y) = 0$ unless $\gamma = \gamma_k$ or $\langle\gamma, \cdot\rangle = 0$, and found agreement with independent results based on blow-up and wall-crossing formulae.

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- The vanishing of $\Omega_\star(\gamma, y)$ is supported by similar arguments about expected dimension, using ad hoc quadratic form $\mathcal{Q}(\gamma)$.
- The computation of D4-D2-D0 indices are insensitive to the value of $\Omega_*(n\delta)$, the BPS index for n D0-branes on X. This value can be fixed by considering D6-brane bound states.

Outline

D6-D4-D2-D0 bound states

- In presence of a non-compact D6-brane, the quiver acquires an additional (ungauged) framing node with f_k arrows $\infty \to \ell$. For $X = K_S$, $f_k = \chi(\mathcal{O}_S, E_k)$.

D6-D4-D2-D0 bound states

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Mozgovoy Reineke '08; Ooguri Yamazaki '08

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- Let $J(Q, W)$ the Jacobian algebra (i.e. the path algebra modded out by relations $\partial_a W = 0$), and Δ_ℓ the set of equivalence classes of paths which start at the vertex ℓ. It admits a partial order with $u \leq v$ if there exists a path w such that $wu \sim v$. The poset Δ_i can be represented as a pyramid or crystal.

D6-D4-D2-D0 bound states

- In the NC chamber, toric fixed points are in one-to-one correspondence with finite ideals $\mathcal{C} \subset \Delta_\ell$, i.e. subsets such that $u \in \mathcal{C}$ whenever $\exists v \in \mathcal{C}$ with $u \leq v$. They can be represented as molten pyramids or molten crystals.

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- Each ideal $\mathcal{C}$ contributes ± 1 to the (unrefined, framed) index $\Omega_{\rm NCDT}(1, d)$ with $d = \sum_{u \in \mathbb{C}} d_u$. The generating series is

$$Z_\ell(x) = \sum_{\mathcal{C} \subset \Delta_\ell} (-1)^{d_\ell + \chi_Q(d,d)}\, x^d$$

Mozgovoy Reineke'08

where $\chi_Q(d, d') = \sum_{a \in Q_0} d_a d'_a - \sum_{a:i \to j} d_a d'_b$ is the Euler form.

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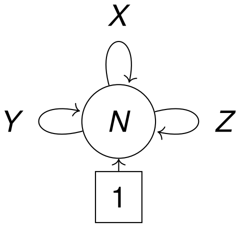
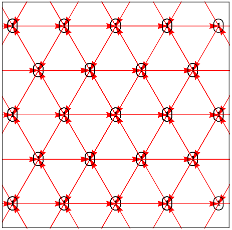
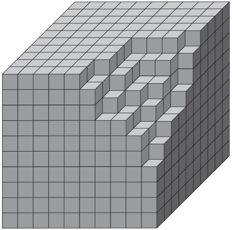
where $\chi_Q(d, d') = \sum_{a \in Q_0} d_a d'_a - \sum_{a: i \to j} d_a d'_b$ is the Euler form.
- Using the flow tree formula for the framed quiver $\tilde{Q}$ with $\Omega_\star(1, d) = 0$ for $d \neq 0$, we can read off the (unrefined, unframed) attractor invariants $\Omega_\star(0, d) = \Omega_\star(d)$.

Example: D6-D0/$\mathbb{C}^3$



$$W = x[y, z]$$





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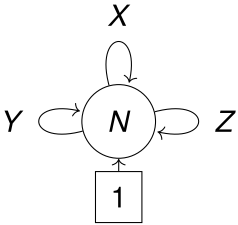

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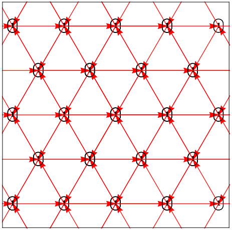

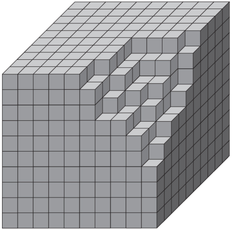

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- The generating function of D6-D0 indices is *[MacMahon 1916]*

$$M(x) = \prod_{k=1}^{\infty} (1-x^k)^{-k} = 1+x+3x^2+6x^3+13x^4+24x^5+48x^6+\dots$$

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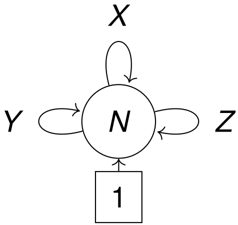

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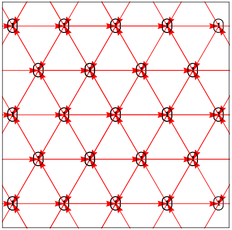

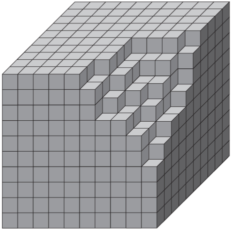

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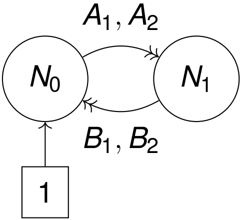
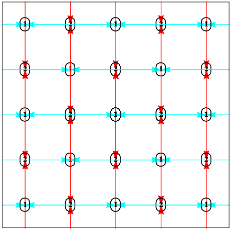
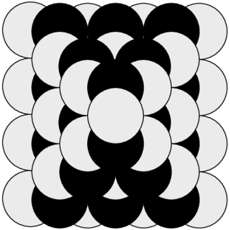
- The unframed, unrefined indices are $\Omega(n) = -1$ for n D0-branes.

Example: D6-D2-D0 on the conifold



$$W = A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1$$





- The generating function of D6-D2-D0 indices is *[Szendroi'07]*

$$\begin{aligned} Z_0 &= M(-x_0 x_1)^2 \prod_{k\geq 1} (1 + x_0^k (-x_1)^{k-1})^k (1 + x_0^k (-x_1)^{k+1})^k \\ &= 1 + x_0 - 2x_0 x_1 + (x_0 x_1^2 - 4x_0^2 x_1) + (8x_0^2 x_1^2 - 2x_0^3 x_1) + \ldots \end{aligned}$$

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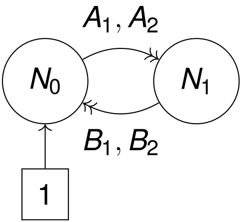

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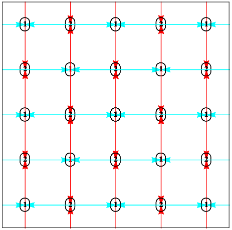

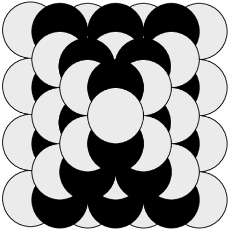

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\end{aligned}
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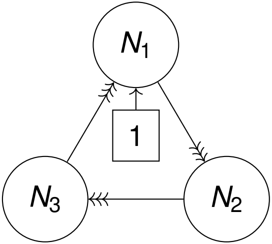
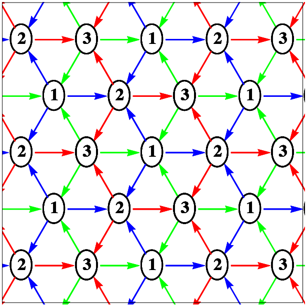
- The non-zero unframed indices are $\Omega(n,n) = -2$, $\Omega(n, n \pm 1) = 1$.

Example: D6-D4-D2-D0 on $\mathbb{C}^3/\mathbb{Z}_3$



$$W = \epsilon_{ijk} \Phi^i_{12} \Phi^j_{23} \Phi^k_{31}$$



- The generating function of D6-D4-D2-D0 indices is

$$Z_1 = 1 + x_1 + 3x_1 x_2 + 3x_1 x_2^2 - 3x_1 x_2 x_3 + 9x_1 x_2^2 x_3 + x_1 x_2^3 - 3x_1^2 x_2 x_3 + \ldots$$

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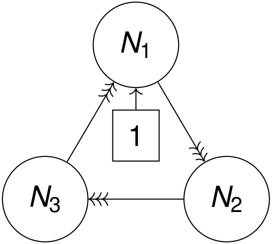

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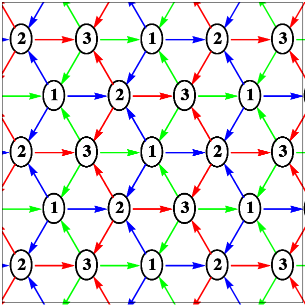

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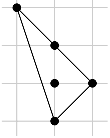
- This is consistent with the vanishing of all attractor indices except $\Omega_\star(n,n,n) = -3 = -\chi(K_{\mathbb{P}_2})$ for n D0-branes.

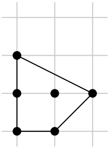
NCDT invariants from attractor indices

- This strategy applies to any brane tiling and allows to determine the (unframed, unrefined) attractor indices by counting molten crystals.
- This confirms our conjecture for Fano surfaces, and indicates that the vanishing of all attractor indices except $\Omega_\star(n\delta) = -\chi_X$ also holds for smooth toric threefolds with more than one compact divisor. Eg: $\mathbb{C}_3/\mathbb{Z}_5$, $Y^{3,2}$, ...
- For singular toric threefolds, such that the boundary of the toric diagram contains lattice points in addition to the corners, one finds $\Omega_\star(d) \neq 0$ for some d in the kernel of $\langle -, - \rangle$. Eg: $\mathbb{F}_2$, PdP_2, $\mathbb{C}^3/\mathbb{Z}_6$, ...

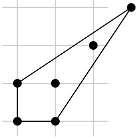
Toric CY threefolds



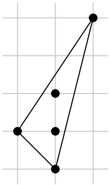
$\mathbb{F}_2$



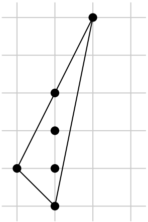
PdP_2



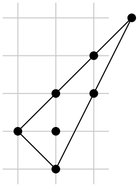
$Y^{3,2}$



$\mathbb{C}^3/\mathbb{Z}_5$



$\mathbb{C}^3/\mathbb{Z}_6(1,1,4)$



$\mathbb{C}^3/\mathbb{Z}_6(1,2,3)$

NCDT invariants from attractor indices

- Assuming the conjecture holds, refined NCDT invariants can be computed for all d once we know $\Omega_\star(n\delta, y)$. The latter can be extracted from the motivic D6-D0 invariants of X:

$$\Omega_\star(n\delta, y) = (-y)^{-3}\,[X] = -b_6/y^3 - b_4/y - y b_2 - y^3 b_0$$

  where b_i are Betti numbers for cohomology with compact support.

  Behrend Bryan Szendroï'09, Manschot BP Sen'10

- For toric CY threefold, $[X]$ can be read off from the toric diagram:

$$\Omega_\star(n\delta, y) = -y^{-3} - (i + b - 3)y^{-1} - i\,y$$

  where i and b are the number of internal and boundary lattice points. For $y = 1$, $\Omega_\star(n\delta) = -(2i + b - 2) = -\chi_X$ is the number of triangles in the toric diagram, by Pick's theorem.

Refined NCDT invariants for $\mathbb{C}^3/\mathbb{Z}_3$

- The generating function of refined framed indices is

$$
\begin{aligned}
Z_1 \;=\;& 1 + x_1 + \left(y^2 + 1 + 1/y^2\right)\left(x_1 x_2 + x_1 x_2^2\right) \\
-\;& \left(y^3 + y + 1/y\right)\left(x_1 x_2 x_3 + x_1^2 x_2 x_3\right) \\
+\;& \left(y^4 + 2y^2 + 3 + 2/y^2 + 1/y^4\right) x_1 x_2^2 x_3 + x_1 x_2^3 \\
-\;& \left(y^5 + y^3 + y + 1/y + 1/y^3 + 1/y^5\right) x_1 x_2^3 x_3 \\
+\;& \left(y^4 + 2y^2 + 3 + 2/y^2 + 1/y^4\right) x_1 x_2^2 x_3^2 \\
-\;& \left(y^5 + 2y^3 + 3y + 2y + 1/y^3\right)\left(x_1^2 x_2^2 x_3 + x_1^2 x_2^3 x_3\right) + \dots
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\end{aligned}
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- These invariants can be confirmed by computing (unframed, refined) DT invariants for trivial stability, and using wall-crossing, or using refined toric localization.

Mozgovoy BP '20; P. Descombes, to appear

Implications for L^2 and single-centered invariants

- The motivic invariants count cohomology classes with compact support on $\mathcal{M}_H(\gamma, \zeta)$, and are usually not invariant under $y \to 1/y$.

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- If so, $\Omega_\star^{L^2}(n\delta, y) = -i(y + 1/y)$ for m D0-branes, where i is the number of internal points, or compact divisors.

Beaujard BP Manschot '20, Duan Ghim Yi '20

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- Single-centered (or pure Higgs) invariants $\Omega_{\rm S}(\gamma,y)$ differ from $\Omega_\star(\gamma,y)$ due to scaling solutions. There is circumstancial evidence that $\Omega_{\rm S}^{L_2}(\gamma,y)=0$ except for the basic D-branes !

Bena et al '12, Lee Wang Yi '12, Manschot BP Sen '12; Mozgovoy BP '20

Outline

Summary and Outlook

- There is overwhelming evidence for the claim that attractor invariants for toric CY3 singularities always vanish, except when they cannot ! Exceptional attractor invariants arise for toric diagrams with lattice points on the boundary. The same seems to hold in non-toric examples.

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- If true, this conjecture gives a new algorithm for computing refined VW invariants and refined NCDT invariants.
- Does this shed light on the mock modular properties of generating series of VW invariants ? How about $\Omega_{\star/\mathrm{S}}(\gamma)$ for compact CY3 ?

Thank you for your attention, and mind the wall !