Attractor indices, brane tilings and crystals

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based on arXiv:2004.14466 with Guillaume Beaujard, Jan Manschot
and arXiv:2012.14358 with Sergey Mozgovoy

Introduction

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- Part of the reason is that $\Omega(\gamma, z)$ depends on the moduli z in an intricate way, due to wall-crossing phenomena associated to BPS bound states with any number of constituents. The moduli space itself receives quantum corrections, unlike in $\mathcal{N} \geq 4$.

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- For D6-D2-D0 bound states for single unit of D6-brane charge at large volume, $\Omega(\gamma, z)$ are the standard Donaldson-Thomas invariants, related to higher-genus GW invariants.

Thomas'99; Maulik Nekrasov Okounkov Panharipande'04

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- In fact, unless the divisor P is irreducible, the generating series of VW invariants is expected to be a (vector-valued) mock modular form, with a precise modular anomaly.

  Alexandrov Banerjee Manschot BP'16-19; Dabholkar Putrov Witten '20

Toric CY3, quivers and brane tilings

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- The nodes of Q corresponds to a basis of absolutely stable branes on X, whose bound states generate the BPS spectrum. For $X = \mathbb{C}^3/\Gamma$, these are the fractional branes; for $X = K_S$, these are elements of an exceptional collection on S.

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- The quiver Q and superpotential W are conveniently summarized by a brane tiling, or equivalently a periodic quiver. The dimension vector d and stability parameters ζ can be deduced from the Chern vector γ and CY moduli z.

Franco Hanany Kenneway Vegh Wecht'05

For toric CY3, attractor indices almost always vanish !

- Since the quiver has oriented loops, the indices $\Omega(\gamma, z) = \Omega(d, \zeta)$ are in general difficult to compute. We claim that quivers associated to toric CY3 are special: the attractor indices

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- More precisely, for $X = K_S$ where S is a Fano surface, we conjecture that $\Omega_*(d) = 0$ unless $d_a = \delta_{a,\ell}$ (corresponding to simple representation, with $\Omega_*(d) = 1$), or $d_a = m$ is the dimension vector associated to m D0-branes.

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- More generally, for toric CY3 singularities, we claim that $\Omega_*(d) = 0$ unless $d_a = \delta_{a,\ell}$ or d lies in (a subspace of) the kernel of the Dirac pairing (i.e. $\langle d, d' \rangle = 0$ for all d'.

- If correct, this conjecture allows to compute the BPS index $\Omega(\gamma, z)$ for any γ, z by using the flow tree formula, or one of its variants.

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  - Other arguments, including computations of refined DT invariants for trivial stability condition.

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Quiver quantum mechanics

- Consider a SUSY quantum mechanics in $0+1$ dimensions, obtained by reducing $\mathcal{N}=1$ gauge theory in $3+1$ dimension, with matter content encoded in a quiver: each node $\ell = 1...K$ represents a $U(N_\ell)$ vector multiplet, each arrow from k to ℓ represents a chiral multiplet in $(N_\ell, \bar{N}_k)$ representation of $U(N_\ell) \times U(N_k)$.

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- The ranks $\{N_\ell\}$ are encoded in a dimension vector $\gamma = \sum N_\ell \gamma_\ell$ in a lattice Γ, endowed with an antisymmetric Dirac pairing $\langle \gamma, \gamma' \rangle = \sum \gamma_{k\ell} N_k N'_\ell$ where $\gamma_{k\ell}$ is the adjacency matrix: the number of arrows from node k to node ℓ counted with sign.

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- In addition, one must specify Fayet-Iliopoulos terms ζ_ℓ such that $(\gamma, \zeta) := \sum_\ell N_\ell \zeta_\ell = 0$, and (in presence of closed oriented loops) a gauge invariant superpotential $W(\phi)$.

Quiver quantum mechanics

- On the Higgs branch, the moduli space of classical SUSY vacua $\mathcal{M}_H(\gamma,\zeta)$ is the set of gauge-inequivalent solutions of the F-term and D-term equations

$$\forall \ell : \sum_{\gamma_{\ell k}>0} \phi^*_{\ell k}\, T^a\, \phi_{\ell k} - \sum_{\gamma_{k\ell}>0} \phi^*_{k\ell}\, T^a\, \phi_{k\ell,\alpha} = \zeta_\ell \,\mathrm{Tr}(T^a)$$

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- 'stable' means that $\mu(\gamma') < \mu(\gamma)$ for any proper subrepresentation with dimension vector $\gamma' < \gamma$, where $\mu(\gamma') = (\sum_\ell \zeta_\ell N'_\ell)/\sum N'_\ell$ is the slope. *[King'94]*

Quiver quantum mechanics

- BPS states correspond to Dolbeault cohomology classes in $H^{p,q}(\mathcal{M}_H,\mathbb{Z})$, counted by the Hodge polynomial

$$\Omega(\gamma,y,t,\zeta)=\sum_{p,q=0}^{2d} h_{p,q}(\mathcal{M}_H(\gamma,\zeta))\,(-y)^{p+q-d}\,t^{p-q}$$

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- The refined BPS index $\Omega(\gamma, y, \zeta) = \Omega(\gamma, y, 1/y, \zeta)$ (the χ_{y^2}-genus). When Dolbeault cohomology is supported in degree $p = q$, it coincides with the Poincaré polynomial. In either case, it reduces to the Euler number in the unrefined limit $y \to 1$.

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- $\Omega(\gamma, y)$ also counts BPS states on Coulomb branch, but that interpretation is subtle due to scaling solutions.

Wall-crossing and attractor indices

- The DT invariants $\Omega(\gamma, y, \zeta)$ jump on hyperplanes where stable representations become semi-stable. For primitive dimension vectors $\gamma_{1,2}$ with Dirac pairing $\gamma_{12} = \langle \gamma_1, \gamma_2 \rangle$,

$$\Delta\Omega(\gamma_1 + \gamma_2, y) = (-1)^{\gamma_{12}} \frac{y^{\gamma_{12}} - y^{-\gamma_{12}}}{y - 1/y} \Omega(\gamma_1, y)\Omega(\gamma_2, y)$$

  across the hyperplane where $\mu(\gamma_1) = \mu(\gamma_2)$. Physically, a two-centered bound state appears/disappears.

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- For more general charges, it is useful to introduce the rational invariants

$$\bar{\Omega}(\gamma, y) = \sum_{m|\gamma} \frac{1}{m} \frac{y - 1/y}{y^m - 1/y^m} \Omega(\gamma/m, y^m)$$

Wall-crossing and attractor indices

- The discontinuity across the hyperplane where $\mu(\gamma_1) = \mu(\gamma_2)$ is then given by a universal wall-crossing formula,

$$\bar{\Omega}(\gamma, y, \zeta_+) = \sum_{\gamma=\sum \alpha_i} \frac{g_{\rm WC}(\{\alpha_i\}, y)}{|{\rm Aut}(\{\alpha_i\})|} \prod_i \bar{\Omega}(\alpha_i, y, \zeta_-)$$

  where $\gamma = M\gamma_1 + N\gamma_2$, $\alpha_i = M_i\gamma_1 + N_i\gamma_2$, and $g_{\rm WC}(\{\alpha_i\}, y)$ is the Poincaré polynomial associated to an Abelian quiver consisting of one vertex v_i for each α_i, and $\langle\alpha_i, \alpha_j\rangle$ arrows from v_i to v_j.

  Konsevitch Soibelman'08, Joyce Song'08; Manschot BP Sen 2010

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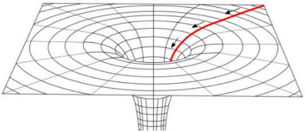
- The WC formula can be derived using localisation in the black hole supersymmetric quantum mechanics. Rational invariants $\bar{\Omega}(\gamma, y)$ arise as effective indices for particles with Boltzmann statistics.

Wall-crossing and attractor indices

- For any dimension vector $\gamma = \sum_\ell N_\ell \gamma_\ell$, there is a particular choice of stabiility parameters

$$\zeta_k^\star(\gamma) = -\gamma_{k\ell} N^\ell = -\langle \gamma_k, \gamma \rangle$$

known as attractor point or self-stability where bound states are ruled out. This is analogous to the attractor point for spherically symmetric black holes in $\mathcal{N} = 2$ supergravity.

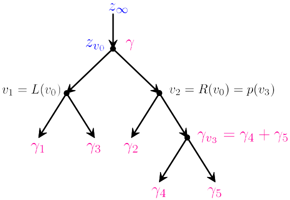


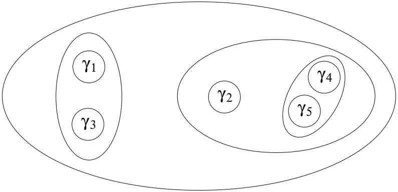
$$\mathrm{Im}[e^{-\mathrm{i}\alpha} X^\Lambda] = q^\Lambda$$
$$\mathrm{Im}[e^{-\mathrm{i}\alpha} F_\Lambda] = p_\Lambda$$
$$\Rightarrow \forall \gamma' \ \mathrm{Im}[e^{-\mathrm{i}\alpha} Z_{\gamma'}] = -\langle \gamma', \gamma \rangle$$

Ferrara Kallosh Strominger'95

Wall-crossing and attractor indices

- The full spectrum can be constructed as bound states of these attractor BPS states, labelled by attractor flow trees:





Denef '00; Denef Green Raugas '01; Denef Moore'07; Manschot '10

Wall-crossing and attractor indices

- The flow tree formula allows to express $\bar\Omega(\gamma, y, \zeta)$ in terms of the attractor indices $\bar\Omega_\star(\gamma_i, y) := \bar\Omega(\alpha_i, y, \zeta^*(\alpha_i))$

$$\bar\Omega(\gamma, y, \zeta) = \sum_{\gamma=\sum \alpha_i} \frac{g_{\rm tr}(\{\alpha_i\}, y, \zeta)}{|{\rm Aut}(\{\alpha_i\})|} \prod_i \bar\Omega_*(\alpha_i, y)$$

Manschot'10, Alexandrov BP '18

where

$$g_{\rm tr}(\{\alpha_i\}, y, \zeta) = \sum_T \prod_{v \in V_T} (-1)^{\gamma_{LR}} \frac{y^{\gamma_{LR}} - y^{-\gamma_{LR}}}{y - 1/y}$$

Here T runs over all possible stable flow trees T ending on the leaves $\alpha_1, \dots, \alpha_n$, v runs over all vertices and $\gamma_{LR} = \langle \gamma_{L(v)}, \gamma_{R(v)} \rangle$.

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- The flow tree formula is combinatorial, and does not require integrating the attractor flow ! It is now a mathematical theorem.

Mozgovoy '20, Argüz Bousseau '21

Outline

Toric CY3 and brane tilings

- Toric CY3 are non-compact CY three-folds which admit an action of $(\mathbb{C}^\times)^3$ having a dense orbit. The category of coherent sheaves $\mathcal{D}(X)$ is isomorphic to the category of representations $\mathcal{D}(Q, W)$ of a quiver with superpotential.

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- The quiver (Q, W) are conveniently summarized by a brane tiling, i.e. a bipartite graph embedded in a two-torus. Tiles correspond to gauge groups, edges to chiral fields, and black/white vertices to monomials in the superpotential. The dual graph is a periodic quiver $\tilde{Q}$ covering Q.

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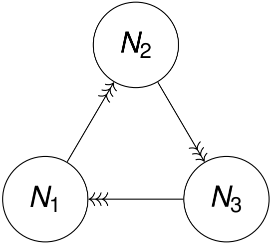
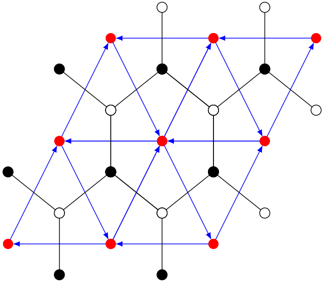
- The same toric CY3 may be described by different tilings/quivers, related by Seiberg duality.

Example: $\mathbb{C}^3/\mathbb{Z}_3 \sim K_{\mathbb{P}^2}$



$$W = \epsilon_{ijk} \Phi^i_{12} \Phi^j_{23} \Phi^k_{31}$$



Outline

Quivers from exceptional collections

- For local surfaces $X = K_S$, a basis of branes on $\mathcal{D}(X)$ (aka tilting sequence) can be constructed from an exceptional collection on S, i.e. an ordered sequence of (virtual) sheaves $(E_1, \ldots, E_r)$ s.t.

$$\begin{aligned} \mathrm{Hom}(E_k, E_k) &= \mathbb{C}\,, \quad \mathrm{Ext}^m_S(E_k, E_k) \quad \forall m > 0 \\ \mathrm{Ext}^m_S(E_k, E_\ell) &= 0 \quad \forall (m \geq 0,\ 1 \leq \ell < k \leq r) \end{aligned}$$

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- Arrows $k \to \ell$ come both from $\mathrm{Ext}^1(E_\ell, E_k)$ and $\mathrm{Ext}^2(E_k, E_\ell)$. The net number is computable from the Euler form on $\mathcal{D}(S)$

$$\chi(E, E') = \sum_{m \geq 0} (-1)^m \dim \mathrm{Ext}^m_S(E, E') = \int_S \mathrm{ch}(E^*)\, \mathrm{ch}(E')\, \mathrm{Td}(S)$$

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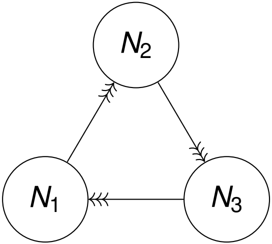
- The dimension vector d and FI parameters ζ can be related to the Chern vector γ and moduli z using $\gamma = \sum N_\ell \gamma_\ell$, $\zeta_\ell = \mathrm{Im}[Z_\gamma \overline{Z_{\gamma_\ell}}]$.

Sheaves on $\mathbb{P}^2$



$$
\begin{aligned}
E_1 &= \mathcal{O} & \gamma_1 &= [1,0,0] \\
E_2 &= \Omega(1)[1] & \gamma_2 &= [-2,1,\tfrac{1}{2}] \\
E_3 &= \mathcal{O}(-1)[2] & \gamma_3 &= [1,-1,\tfrac{1}{2}]
\end{aligned}
$$

$$
\chi(E_k, E_\ell) = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & -3 & 1 \end{pmatrix}
$$

[Le Potier'94]

Dimension vector: ($\propto (1,1,1)$ for D0-branes)

$$
(N_1, N_2, N_3) = -\left(\tfrac{3}{2}c_1 + \mathrm{ch}_2 + N, \tfrac{1}{2}c_1 + \mathrm{ch}_2, -\tfrac{1}{2}c_1 + \mathrm{ch}_2\right)
$$

For canonical polarization $J = \rho c_1(S)$ with $\rho \gg 1$,

$$
\zeta = 3\rho\,(N_2 - N_3, N_3 - N_1, N_1 - N_2) + \left(-\frac{N_2+N_3}{2}, \frac{N_1+3N_3}{2}, \frac{N_1-3N_2}{2}\right)
$$

Canonical vs. attractor chamber

- For any $X = K_S$, the canonical chamber $J = \rho\, c_1(S)$ in the large volume limit translates into the anti-attractor chamber,

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$$d_{\mathbb{C}} = \sum_{a \notin I} N_k N_\ell - \sum_{a \in I} N_k N_\ell - \sum N_\ell^2 + 1 = 1 - \chi(E, E)$$

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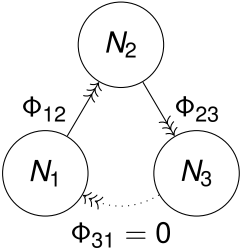
- In contrast, in the attractor chamber $\zeta_k = -\rho \sum_\ell \gamma_{k\ell} N_\ell$, the expected dimension is always negative, unless the dimension vector is one of the basis vectors γ_ℓ, or lies in the kernel of $\langle -, \rangle$.

Sheaves on $\mathbb{P}^2$



$$N_1 = -\left(\tfrac{3}{2}c_1 + \mathrm{ch}_2 + N\right)$$
$$N_2 = -\left(\tfrac{1}{2}c_1 + \mathrm{ch}_2\right)$$
$$N_3 = -\left(-\tfrac{1}{2}c_1 + \mathrm{ch}_2\right)$$

• In canonical (anti-attractor chamber), the expected dimension is positive at large c_2,

$$\begin{aligned} d_{\mathbb{C}} &= 3(N_1 N_2 + N_2 N_3 - N_3 N_1) - N_1^2 - N_2^2 - N_3^2 + 1 \\ &= c_1^2 - 2\,\mathrm{rk}\,\mathrm{ch}_2 - \mathrm{rk}^2 + 1 \end{aligned}$$

This requires $\zeta_1 \geq 0, \zeta_3 \leq 0$ hence $-N \leq c_1 \leq 0$.

- In attractor chamber $\zeta^\star = 3\,(N_2 - N_3, N_3 - N_1, N_1 - N_2)$, the expected dimension is almost always negative:

$$d^\star_{\mathbb{C}} = 1 - \mathcal{Q}(\gamma) + \begin{cases} \frac{2}{3}N_3\zeta^\star_3 - \frac{2}{3}N_1\zeta^\star_1 & \zeta^\star_1 \geq 0, \zeta^\star_3 \leq 0 \quad (\Phi_{31} = 0) \\ \frac{2}{3}N_1\zeta^\star_1 - \frac{2}{3}N_2\zeta^\star_2 & \zeta^\star_2 \geq 0, \zeta^\star_1 \leq 0 \quad (\Phi_{12} = 0) \\ \frac{2}{3}N_2\zeta^\star_2 - \frac{2}{3}N_3\zeta^\star_3 & \zeta^\star_3 \geq 0, \zeta^\star_2 \leq 0 \quad (\Phi_{23} = 0) \end{cases}$$

$$\mathcal{Q}(\gamma) = \tfrac{1}{2}(N_1 - N_2)^2 + \tfrac{1}{2}(N_2 - N_3)^2 + \tfrac{1}{2}(N_3 - N_1)^2$$

hence $d^*_{\mathbb{C}} < 0$ unless $\gamma \in \{(1,0,0),(0,1,0),(0,0,1),(n,n,n)\}$.

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- We conjecture that $\Omega_\star(\gamma) = 0$ except in those cases. We set $\Omega_\star(1,0,0) = \Omega_\star(0,1,0) = \Omega_\star(0,0,1) = 1$, The index $\Omega_\star(n,n,n)$, corresponding to n D0-branes will be specified later.

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- This argument is not (yet) a mathematical proof.

VW invariants on $\mathbb{P}^2$

- Using the flow tree formula, and assuming the conjecture, we find that the index in canonical chamber agrees with VW invariants on $\mathbb{P}^2$ previously computed using blow-up/wall-crossing formulae !

Goettsche'90, Klyachko'91, Yoshioka'94, Manschot'11-14

$[N; c_1; c_2]$	(N_1, N_2, N_3)	$\Omega(\gamma, -\zeta^\star(\gamma))$
$[1; 0; 2]$	$(1, 2, 2)$	$y^4 + 2y^2 + 3 + \dots$
$[1; 0; 3]$	$(2, 3, 3)$	$y^6 + 2y^4 + 5y^2 + 6 + \dots$
$[2; 0; 3]$	$(1, 3, 3)$	$-y^9 - 2y^7 - 4y^5 - 6y^3 - 6y - \dots$
$[2; -1; 2]$	$(1, 2, 1)$	$y^4 + 2y^2 + 3 + \dots$
$[2; -1; 3]$	$(2, 3, 2)$	$y^8 + 2y^6 + 6y^4 + 9y^2 + 12 + \dots$
$[3; -1; 3]$	$(1, 3, 2)$	$y^8 + 2y^6 + 5y^4 + 8y^2 + 10 + \dots$
$[4; -2; 4]$	$(1, 3, 1)$	$y^5 + y^3 + y + \dots$

Attractor invariants for Fano surfaces

- We conjecture that the vanishing of attractor invariants holds for any CY threefold $X = K_S$ where S is a Fano surface. This includes the toric cases $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $S = dP_{k\leq 3}$, but also the non-toric del Pezzo surfaces $dP_{4\leq k\leq 8}$.

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- For those cases, we have computed VW invariants using the flow tree formula, under the assumption that $\Omega_\star(\gamma, y) = 0$ unless $\gamma = \gamma_k$ or $\langle \gamma, \cdot \rangle = 0$, and found agreement with independent results based on blow-up and wall-crossing formulae.

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- The vanishing of $\Omega_\star(\gamma, y)$ is supported by similar arguments about expected dimension, using ad hoc quadratic form $\mathcal{Q}(\gamma)$.
- The computation of D4-D2-D0 indices are insensitive to the value of $\Omega_*(n\delta)$, the BPS index for n D0-branes on X. This value can be fixed by considering D6-brane bound states.

Outline

D6-D4-D2-D0 bound states

- In presence of a non-compact D6-brane, the quiver acquires an additional (ungauged) framing node with f_k arrows $\infty \to \ell$. For $X = K_S$, $f_k = \chi(\mathcal{O}_S, E_k)$.

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- For simplicity we assume a single framing arrow, $f_k = \delta_{k,\ell}$. The framed DT invariants $\Omega(1, d)$ in the non-commutative (NC) chamber $\zeta_\infty > 0, \zeta_k < 0$ can be computed by torus localization.

Mozgovoy Reineke'08

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Mozgovoy Reineke'08

- Let $J(Q, W)$ the Jacobian algebra (i.e. the path algebra modded out by relations $\partial_a W = 0$), and Δ_ℓ the set of equivalence classes of paths which start at the vertex ℓ. It admits a partial order with $u \leq v$ if there exists a path w such that $wu \sim v$. Δ_i can be represented as a pyramid or crystal.

D6-D4-D2-D0 bound states

- In the NC chamber, toric fixed points are in one-to-one correspondence with finite ideals $\mathcal{C} \subset \Delta_\ell$, i.e. subsets such that $u \in \mathcal{C}$ whenever $\exists v \in \mathcal{C}$ with $u \leq v$. They can be represented as molten pyramids or molten crystals.

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$$Z_\ell(x) = \sum_{\mathcal{C} \subset \Delta_\ell} (-1)^{d_\ell + \chi_Q(d,d)}\, x^d$$

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Mozgovoy Reineke'08

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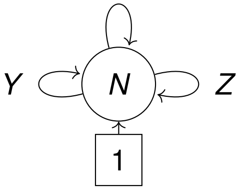
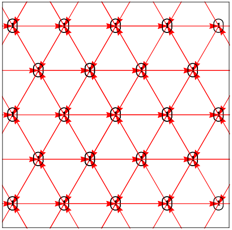
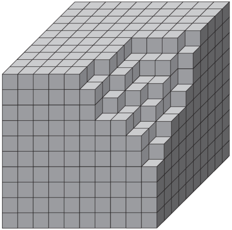
- Using the flow tree formula for quiver $\tilde{Q}$ with $\Omega_\star(1,d) = 0$ for $d \neq 0$, we can read off the (unrefined, unframed) attractor invariants $\Omega_\star(0,d)$.

Example: D6-D0/$\mathbb{C}^3$

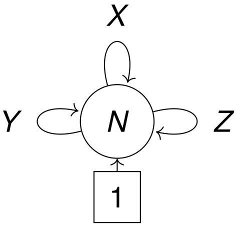


$$W = x[y, z]$$





- The Jacobian algebra is $J(Q, W) = \mathbb{C}[x, y, z]$. Ideals correspond to plane partitions, or molten configurations of the crystal $\mathbb{N}^3$.



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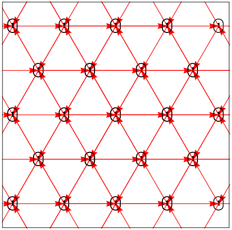

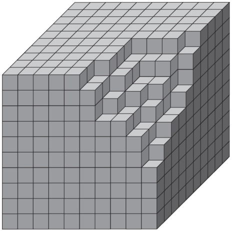

- The Jacobian algebra is $J(Q,W) = \mathbb{C}[x,y,z]$. Ideals correspond to plane partitions, or molten configurations of the crystal $\mathbb{N}^3$.
- The generating function of D6-D0 indices is *[Mac-Mahon 1916]*

$$M(x) = \prod_{k=1}^{\infty} (1-x^k)^{-k} = 1+x+3x^2+6x^3+13x^4+24x^5+48x^6+\dots$$

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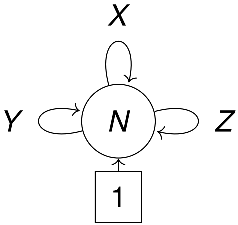

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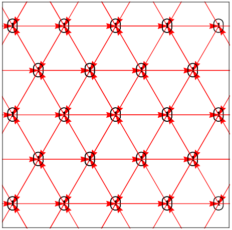

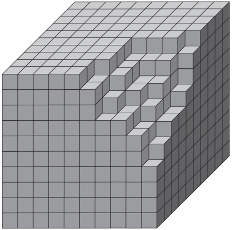

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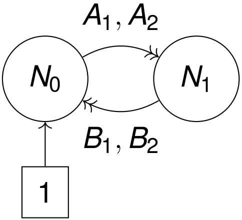
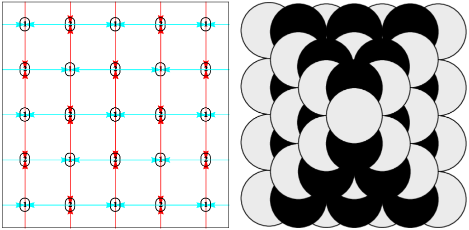
- The unframed, unrefined indices are $\Omega(N) = -1$ for N D0-branes.

Example: D6-D2-D0 on the conifold

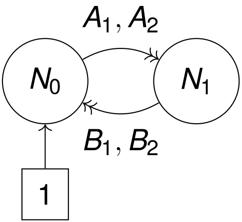


$$W = A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1$$

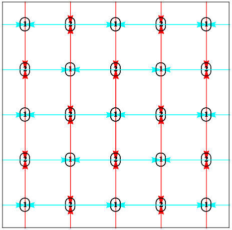


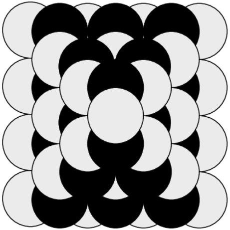
- The generating function of D6-D2-D0 indices is *[Szendroi'07]*

$$\begin{aligned} Z_0 &= M(-x_0 x_1)^2 \prod_{k\geq 1} (1 + x_0^k (-x_1)^{k-1})^k (1 + x_0^k (-x_1)^{k+1})^k \\ &= 1 + x_0 - 2x_0 x_1 + (x_0 x_1^2 - 4x_0^2 x_1) + (8x_0^2 x_1^2 - 2x_0^3 x_1) + \dots \end{aligned}$$



$$W = A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1$$





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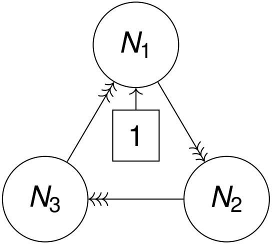
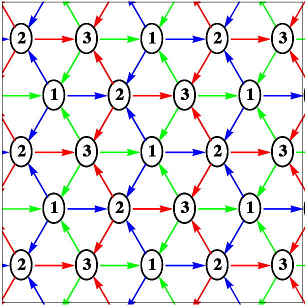
- The non-zero unframed indices are $\Omega(n,n) = -2$, $\Omega(n, n\pm 1) = 1$.

Example: D6-D4-D2-D0 on $\mathbb{C}^3/\mathbb{Z}_3$



$$W = \epsilon_{ijk}\Phi^i_{12}\Phi^j_{23}\Phi^k_{31}$$



- The generating function of D6-D4-D2-D0 indices is

$$Z_1 = 1 + x_1 + 3x_1x_2 + 3x_1x_2^2 - 3x_1x_2x_3 + 9x_1x_2^2x_3 + x_1x_2^3 - 3x_1^2x_2x_3 + \ldots$$

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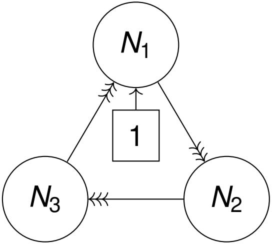

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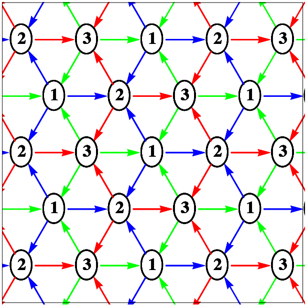

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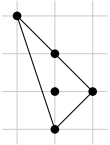
- This is consistent with the vanishing of all attractor indices except $\Omega_\star(n,n,n) = -3 = -\chi(K_{\mathbb{P}_2})$ for n D0-branes.

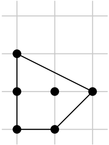
NCDT invariants from attractor indices

- This strategy applies to any brane tiling and allows to determine the (unframed, unrefined) attractor indices by counting molten crystals.
- This confirms our conjecture for Fano surfaces, and indicates that the vanishing of all attractor indices except $\Omega_\star(n\delta) = -\chi(X)$ also holds for smooth toric threefolds with more than one compact divisor. Eg: $\mathbb{C}_3/\mathbb{Z}_5$, $Y^{3,2}$, ...
- For singular toric threefolds, such that the boundary of the toric diagram contains lattice points in addition to the corners, one finds $\Omega_\star(d) \neq 0$ for some d in the kernel of $\langle -, - \rangle$. Eg: $\mathbb{F}_2$, PdP_2, $\mathbb{C}^3/\mathbb{Z}_6$, ...

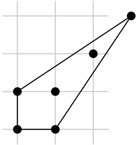
Toric CY threefolds



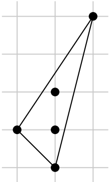
$\mathbb{F}_2$



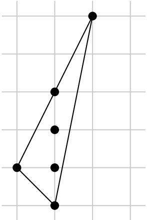
PdP_2



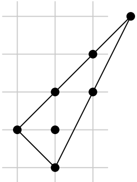
$Y^{3,2}$



$\mathbb{C}^3/\mathbb{Z}_5$



$\mathbb{C}^3/\mathbb{Z}_6(1,1,4)$



$\mathbb{C}^3/\mathbb{Z}_6(1,2,3)$

NCDT invariants from attractor indices

- Assuming the conjecture holds, refined NCDT invariants can be computed for all d once we know $\Omega_\star(n\delta, y)$. The latter can be extracted from the motivic D6-D0 invariants of X:

$$\Omega_\star(n\delta, y) = (-y)^{-3}\,[\mathcal{X}] = -b_6/y^3 - b_4/y - yb_2 - y^3 b_0$$

where b_i are Betti numbers for cohomology with compact support.

Behrend Bryan Szendroï'09, Manschot BP Sen'10

- For toric CY threefold, $[X]$ can be read off from the toric diagram:

$$\Omega_\star(n\delta, y) = -y^{-3} - (i + b - 3)y^{-1} - i\,y$$

where i and b are the number of internal and boundary lattice points. For $y = 1$, $\Omega_\star(n\delta) = -(2i + b - 2) = -\chi(\mathcal{X})$ is the number of triangles in the toric diagram, by Pick's theorem.

Refined NCDT invariants for $\mathbb{C}^3/\mathbb{Z}_3$

- The generating function of refined framed indices is

$$\begin{aligned}
Z_1 \;=\; & 1 + x_1 + \left(y^2 + 1 + 1/y^2\right)\left(x_1 x_2 + x_1 x_2^2\right) \\
- & \left(y^3 + y + 1/y\right)\left(x_1 x_2 x_3 + x_1^2 x_2 x_3\right) \\
+ & \left(y^4 + 2y^2 + 3 + 2/y^2 + 1/y^4\right) x_1 x_2^2 x_3 + x_1 x_2^3 \\
- & \left(y^5 + y^3 + y + 1/y + 1/y^3 + 1/y^5\right) x_1 x_2^3 x_3 \\
+ & \left(y^4 + 2y^2 + 3 + 2/y^2 + 1/y^4\right) x_1 x_2^2 x_3^2 \\
- & \left(y^5 + 2y^3 + 3y + 2y + 1/y^3\right)(x_1^2 x_2^2 x_3 + x_1^2 x_2^3 x_3) + \dots
\end{aligned}$$

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&+ \left(y^4 + 2y^2 + 3 + 2/y^2 + 1/y^4\right) x_1 x_2^2 x_3 + x_1 x_2^3 \\
&- \left(y^5 + y^3 + y + 1/y + 1/y^3 + 1/y^5\right) x_1 x_2^3 x_3 \\
&+ \left(y^4 + 2y^2 + 3 + 2/y^2 + 1/y^4\right) x_1 x_2^2 x_3^2 \\
&- \left(y^5 + 2y^3 + 3y + 2y + 1/y^3\right)(x_1^2 x_2^2 x_3 + x_1^2 x_2^3 x_3) + \ldots
\end{aligned}
$$

- These invariants can be confirmed by computing (unframed, refined) DT invariants for trivial stability, and using wall-crossing.

Mozgovoy BP'20

Outline

Summary and Outlook

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- Does this shed light on the mock modular properties of generating series of VW invariants ? How about compact CY3 ?

Thank you for your attention, and mind the wall !