Black Hole Degeneracies, Topological Strings and Quantum Attractors

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Lecture 4: The Quantum Attractor Flow

based on Ooguri Vafa Verlinde 0502211
BP 0506228
Gunaydin, Neitzke, BP and Waldron 0512296
and work in progress

The OSV Conjecture

- Based on the observation that the Legendre transform of the BHW entropy has a simple relation to the topological string amplitude, Ooguri, Strominger and Vafa (OSV) have proposed a simple relation between micro-canonical degeneracies $\Omega(p^I, q_I)$ and the topological string amplitude:

$$\Omega(p^I, q_I) \sim \int d\phi^I |\Psi_{top}(p^I + i\phi^I)|^2 e^{\phi^I q_I} \qquad (*)$$

where $\Psi_{top}(X^I) = \exp\left(\frac{i\pi}{2} F(X^I)\right)$ is the topological wave function. Equivalently,

$$\sum_{q_I \in \Lambda_{el}} \Omega(p^I, q_I) e^{-\phi^I q_I} \sim \sum_{k^I \in \Lambda^*_{el}} \Psi^*_{top}(p^I + k^I + i\phi^I) \Psi_{top}(p^I - k^I + i\phi^I) \qquad (**)$$

- The $\sim$ sign in (**) allegedly denotes an equality to all orders in an expansion at large charges $(\lambda p^I, \lambda q_I)$, $\lambda \to \infty$. A non-perturbative generalization might hold upon completing the perturbative topological string amplitude and specifying a contour.
- This conjecture has many problems: symplectic invariance, holomorphic anomalies, . . . but does work amazingly well in some cases.

OSV conjecture and quantum mechanics

- Performing a Wick rotation $\phi^I = i\chi^I$, (*) becomes

$$\Omega(p^I, q_I) \sim \int d\chi^I \Psi^*_{top}(p^I + \chi^I)\Psi_{top}(p^I - \chi^I)e^{i\phi^I q_I}$$

This is recognized as the Wigner distribution associated to wave function $\Psi_{top}(p^I)$. In ordinary quantum mechanics, this provides a semi-classical description of the state Ψ_{top} in terms of a probability density $W(p,q)$ on phase space (in general non-positive).

- Defining

$$\Psi_{p,q}(\chi) := e^{iq\chi}\Psi_{top}(\chi - p) := V_{p,q} \cdot \Psi_{top}(\chi)$$

this can be rewritten even more suggestively as

$$\Omega(p,q) \sim \int d\chi \; \Psi^*_{p,q}(\chi) \; \Psi_{p,q}(\chi)$$

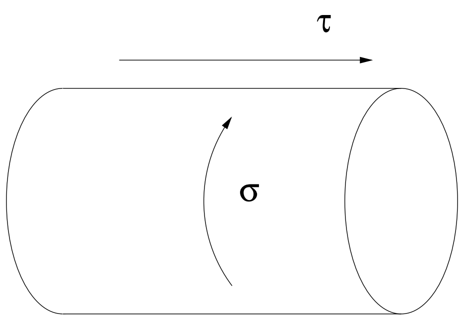
where the dependence on p, q is absorbed in Ψ: This is an overlap between two wave functions. But of what physical system ?

OSV conjecture and channel duality

- This is reminiscent of the familiar open/closed duality for conformal field theory on the cylinder,

$$\mathrm{Tr} e^{-\pi t H_{open}} = \langle B | e^{-\frac{\pi}{t} H_{closed}} | B \rangle$$

where H_{open} is the Hamiltonian generating translations in σ, H_{closed} is the Hamiltonian describing translations in τ, and $|B\rangle$ is the boundary state which encodes the boundary conditions at $\tau = 0, t$

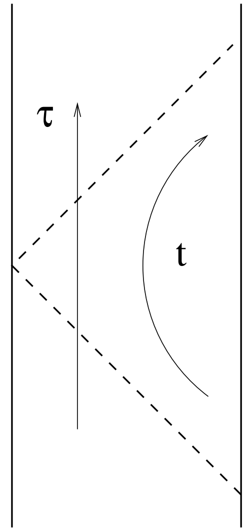


- In this analogy, $\Omega(p, q)$ is the trace of the open string Hamiltonian in the Hilbert space with charge (p, q), and $\Psi_{p,q}$ is the closed string boundary state. For the analogy to hold, both H_{open} and H_{closed} should vanish.

Ooguri Vafa Verlinde

Topological amplitude and quantum radial flow

- Indeed, the near-horizon geometry $AdS_2 \times S^2$ has the topology of a cylinder, and can in principle be quantized in two ways:



(global or Poincaré) time	$\leftrightarrow$	Conformal Quantum Mechanics
Radial quantization	$\leftrightarrow$	Quantum Attractor Flow

(Both Hamiltonians vanish due to the diffeomorphism invariance.)

The equality between the two channels is a mini-version of AdS/CFT.

Ooguri Vafa Verlinde;Dijkgraaf Gopakumar Ooguri Vafa; Gukov Saraikin Vafa

- In this interpretation, the topological amplitude is understood as a particular wave function for the radial attractor flow, in a "mini-superspace" approximation where only spherically symmetric geometries are retained.

Radial BH quantization and the Universe wave function

- Radial quantization of black holes is not a new idea: in fact much work was done on this problem in the gr-qc community, but yielded little insight on the nature of black hole micro-states.

Cavaglia de Alfaro Filippov; Kuchar; Thiemann Kastrup; Breitenlohner Hellmann

- One novelty here is that one works in a SUSY context, for which the "mini-superspace" truncation to spherically symmetric geometries, and omission of D-term interactions, has (perhaps) some chance of being exact.
- Furthermore, the idea of holography supports the idea that the spectrum of the global time Hamiltonian can be reconstructed from the radial wave functions.
- Further interest arises from the fact that the black hole attractor equations are very similar to those that determine supersymmetric vacua in flux compactifications. Upon double analytic continuation, the black hole wave function can (perhaps) be interpreted as the Hartle-Hawking wave function of the Universe.
- Q: is there a physical principle that picks out Ψ_{top} from the infinite dimensional SUSY Hilbert space ?

Outline of the lecture

- Our goal is to try and clarify these ideas, by considering situations with higher symmetry: $N = 8$ and $N = 4$ SUGRA, or "very special" $N = 2$ SUGRA. The complexity of CY geometry is jettisoned in favor of representation theory.
- For this we shall reinterpret the attractor equations for 4D black holes as (BPS) geodesic motion on the scalar manifold $\mathcal{M}_3^*$ of the 3D SUGRA obtained by reducing 4D SUGRA along the time direction.

Breitenlohner Gibbons Maison, Gutperle Spalinski

- This geodesic motion is then quantized by replacing classical trajectories by functions on $\mathcal{M}_3^*$. BPS trajectories quantize into special (e.g. holomorphic) functions. When $\mathcal{M}_3^* = G_3/K_3^*$ is symmetric, the (BPS) Hilbert space may be understood in terms of (unusually small) irreps of G_3.

Gross Wallach; Kazhdan BP Waldron; Gunaydin Koepsell Nicolai

- Our main message is that, beyond the expected 4D U-duality symmetry, under which black hole degeneracies ought to be invariant, there is a larger "spectrum generating" symmetry G_3, the 3D U-duality group , which underlies the black hole wave function. Exact degeneracies should be expressed in terms of Fourier coefficients of automorphic forms for $G_3(\mathbb{Z})$.
- Warning: work in progress, many loose ends remain.

Plan of the lecture

- Attractor flow and geodesic motion
- Very special supergravities and the quasi-conformal representation
- The quantum attractor flow
- The automorphic black hole wave function
- Open problems

Attractor flow and KK* reduction

- Stationary solutions in 4D can be parameterized in the form

$$ds_4^2 = -e^{2U}(dt+\omega)^2 + e^{-2U} ds_3^2 \,, \quad A_4^I = \zeta^I dt + A_3^I$$

  where $ds_3, U, \omega, A_3^I, \zeta^I$ are independent of time. The D=3+1 theory reduces to a field theory in 3 Euclidean dimensions.

- In contrast to the usual KK ansatz,

$$ds_4^2 = e^{2U}(dy+\omega)^2 + e^{-2U} ds_{2,1}^2 \,, \quad A_4^I = \zeta^I dy + A_3^I$$

  where the fields are independent of y, we reduce on a time-like direction.

- For the usual KK reduction to 2+1D, the one-forms (A_3^I, ω) can be dualized into pseudo-scalars $(\tilde{\zeta}_I, a)$. The 4D Einstein-Maxwell equations reduce to 3D gravity + scalars living in a Riemannian space $\mathcal{M}_3$.

$KK \to KK^* + \nu$

- The KK* reduction is simply related to the KK reduction by letting $(\zeta^I, \tilde{\zeta}_I) \longrightarrow i(\zeta^I, \tilde{\zeta}_I)$. As a result, the scalar fields live in a pseudo-Riemannian space $\mathcal{M}_3^*$, with non-positive definite signature.

Breitenlohner Gibbons Maison; Hull Julia

- $\mathcal{M}_3^*$ always has $2n+2$ isometries corresponding to the gauge symmetries of $A^I, \tilde{A}_I, \omega$, as well as rescalings of time t. The Killing vector fields satisfy the algebra

$$[p^I, q_J] = 2\delta^I_J\, k\ , \quad [m, p^I] = p^I,\ [m, q_I] = q_I\ , [m, k] = 2k$$

- As we shall see shortly, black hole solutions correspond to geodesic motion on $\mathcal{M}_3^*$; as the notation suggests, the conserved charges associated to these isometries will be identified to electric and magnetic charges, NUT charge and ADM mass.

c-map and c*-map

- The reduction of tree-level $4D$ $N = 2$ SUGRA coupled to vector multiplets to 2+1 dimensions is well studied [hypers go along for the ride]: the Riemannian space is a quaternionic-Kähler space, entirely determined by the tree-level prepotential in 4 dimensions:

$$ds^2 = 2(dU)^2 + g_{i\bar{j}}(z,\bar{z})dz^i dz^{\bar{j}} + \frac{1}{2}e^{-4U}\left(da + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)^2$$
$$-e^{-2U}\left[(\mathrm{Im}\mathcal{N})_{IJ}d\zeta^I d\zeta^J + (\mathrm{Im}\mathcal{N}^{-1})^{IJ}\left(d\tilde{\zeta}_I + (\mathrm{Re}\mathcal{N})_{IK}d\zeta^K\right)\left(d\tilde{\zeta}_J + (\mathrm{Re}\mathcal{N})_{JL}d\zeta^L\right)\right]$$

where

$$\mathcal{N}_{IJ} = \bar{\tau}_{IJ} + 2i\frac{(\mathrm{Im}\tau_{IK}X^K)(\mathrm{Im}\tau_{JL}X^L)}{X^K\mathrm{Im}\tau_{KL}X^L}\,, \quad \tau_{IJ} := \partial_{IJ}F$$

- This is known as the c-map of the original special Kähler manifold. This construction originally arose in a purely 4D context, in relation with mirror symmetry.

Ferrara Sabharwal; de Wit Van Proyen Vanderseypen

- The manifold $\mathcal{M}_3^*$ obtained by analytic continuation $(\zeta^I, \tilde{\zeta}_I) \to i(\zeta^I, \tilde{\zeta}_I)$ is sometimes called “para-quaternionic-Kahler manifold”.

Quaternionic-Kähler geometry

- Recall that a quaternionic-Kähler space is a manifold with special holonomy $USp(2) \times USp(2n) \subset SO(4n)$. It admits three almost complex structures J^i satisfying the quaternion algebra,

$$J^i \cdot J^j = -\delta^{ij} + \epsilon^{ijk} J^k$$

The associated 2-forms $\Omega^i(X,Y) = g(X, J^i Y)$ are covariantly constant with respect to a $USp(2) = SU(2)$ connection p^i whose curvature is proportional to Ω^i,

$$d\Omega^i + \epsilon^{ijk} p^j \wedge \Omega^k = 0 \;, \quad dp^i + \epsilon^{ijk} p^j \wedge p^k = -i\Omega^i$$

- The $USp(2) \times USp(2n)$ connection $p+q$ may be obtained from a covariantly constant quaternionic viel-bein $V^{\alpha}\Gamma$, $\alpha = 1, 2$, $\Gamma = 1, .., 2n$ such that

$$\Omega^i = \epsilon_{\alpha\beta} \, (\sigma^i)_\gamma^\beta \rho_{\Gamma\Gamma'} V^{\alpha\Gamma} \wedge V^{\gamma\Gamma'} \;, \quad ds^2 = \epsilon_{\alpha\beta} \rho_{\Gamma\Gamma'} V^{\alpha\Gamma} \otimes V^{\beta\Gamma'} \;, \quad (d+\Omega)V = 0$$

- The quaternionic viel-bein controls the fermionic SUSY variations,

$$\delta\chi^\Gamma = V_i^{\alpha\Gamma} \partial_\mu \phi^i \sigma_\alpha^{\mu\beta} \epsilon_\beta + O(\chi^2)$$

Bagger Witten

The c-map is quaternionic-Kähler

- For later reference, let us record the quaternionic viel-bein for the $c-map$,

$$V^{\alpha\Gamma} = \begin{pmatrix} u & v \\ e^A & E^A \\ -\bar{v} & \bar{u} \\ -\bar{E}^A & \bar{e}^A \end{pmatrix}$$

where $e^A = e_i^A dz^i$ is a viel-bein of the Special Kähler manifold, $e_i^A \bar{e}_{A\bar{j}} = g_{i\bar{j}}$, and

$$\begin{aligned} u &= e^{K/2-U} X^I \left(d\tilde{\zeta}_I + \mathcal{N}_{IJ} d\zeta^J \right) \\ v &= -dU + \frac{i}{2} e^{-2U} \left(da + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}^I d\zeta_I \right) \\ E^A &= e^{-U} e_i^A g^{i\bar{j}} \bar{f}_{\bar{j}}^I \left(d\tilde{\zeta}_I + \mathcal{N}_{IJ} d\zeta^J \right) \end{aligned}$$

- For the c^*-map, the same formalism goes through but reality conditions change: $\bar{u} = -u^*$, $\bar{E} = -E^*$

Attractor flow and geodesic motion

- Now, restrict to spherically symmetric solutions, $ds_3^2 = N^2(\rho)d\rho^2 + r^2(\rho)d\Omega_2^2$. The sigma-model action becomes, up to a total derivative (g_{ij} is the metric on $\mathcal{M}_3^*$):

$$S = \int d\rho \left[\frac{N}{2} + \frac{1}{2N} \left(\dot{r}^2 - r^2 g_{ij} \dot{\phi}^i \dot{\phi}^j \right) \right]$$

- The lapse N can be set to 1, but it imposes the Hamiltonian constraint

$$H_{WDW} = (p_r)^2 - \frac{1}{r^2} g^{ij} p_i p_j - 1 \equiv 0$$

  Solutions are thus massive geodesics on the cone $\mathbb{R}^+ \times \mathcal{M}_3^*$. This separates into geodesic motion on $\mathcal{M}_3^*$, times motion along r.

- BPS states need to have flat 3D slices, so we may set set $N = 1, r = \rho$ from the outset: A necessary condition for SUSY is therefore that geodesics be light-like.
- Keeping the variable r is important for defining observables such as the horizon area, $A_H = e^{-2U} r^2|_{U \to -\infty}$ and ADM mass $M_{ADM} = r(e^{2U} - 1)|_{U \to 0}$.

Geodesic motion and conserved charges

- The isometries of $\mathcal{M}_3$ imply conserved Noether charges, whose Poisson bracket reflect the Lie algebra of the isometries. In particular, the electric and magnetic charges satisfy an Heisenberg algebra, whose center is the NUT charge k:

$$[p^I, q_J] = 2\delta^I_J k$$

  Note that the ADM mass does NOT Poisson-commute with (p, q, k).
- If $k \neq 0$, the 4D metric contains an off-diagonal term,

$$ds_4^2 = -e^{2U}(dt + k\cos\theta d\phi)^2 + e^{-2U}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2]$$

  This implies that the metric has CTC's at infinity.
- Bona fide 4D black holes need to have $k = 0$: this is a kind of classical limit. This meshes well with the OSV conjecture, which identifies $\Omega(p, q)$ as the Wigner function of the quantum wave function Ψ...Keeping $k \neq 0$ allows to greatly extend the symmetry.

Geodesic flow on special quaternionic Kahler manifolds

- Let us now reproduce the attractor flow equations of BPS black holes in $N = 2$ SUGRA from geodesic flow $\mathcal{M}_3^* = c^*$-map$(\mathcal{M}_4)$. The conserved charges corresponding to the shift isometries are

$$\begin{aligned} q_I &= -2e^{-2U}\left[(\mathrm{Im}\mathcal{N})_{IJ}d\zeta^J + (\mathrm{Re}\mathcal{N})_{IJ}(Im\mathcal{N}^{-1})^{JL}\left(d\tilde{\zeta}_L + (\mathrm{Re}\mathcal{N})_{LM}d\zeta^M\right)\right] + 2\,k\tilde{\zeta}_I \\ p^I &= -2e^{-2U}(\mathrm{Im}\mathcal{N}^{-1})^{IL}\left(d\tilde{\zeta}_L + (\mathrm{Re}\mathcal{N})_{LM}d\zeta^M\right) - 2\,k\zeta^I \\ k &= e^{-4U}\left(da + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}^I d\zeta_I\right) \end{aligned}$$

- This can be inverted to express $d\zeta^I, d\tilde{\zeta}_I, da$ in terms of q_I, p^I, k, hence

$$u = -\frac{i}{2}e^{K/2+U}X^I\left[q_I - 2k\tilde{\zeta}_I - \mathcal{N}_{IJ}(p^J + 2k\zeta^J)\right]\ ,\quad v = -dU + \frac{i}{2}e^{2U}k$$

$$e^A = e_i^A\,dz^i\ ,\quad E^A = -\frac{i}{2}e^U e^{Ai}g^{i\bar{j}}\bar{f}_{\bar{j}}^I\left[q_I - 2k\tilde{\zeta}_I - \mathcal{N}_{IJ}(p^J + 2k\zeta^J)\right]$$

SUSY geodesic flow and generalized attractor equations

- The BH solution preserves 1/2 SUSY iff there exists $\epsilon_\alpha \neq 0$ such that

$$\delta\chi^\Gamma = V_\mu^{\alpha\Gamma}\sigma_\alpha^{\mu\beta}\epsilon_\beta = V^{\alpha\Gamma}\tilde{\epsilon}_\alpha = 0$$

Equivalently, the rectangular matrix V should have a zero eigenvector $(1, \lambda)$:

$$\begin{aligned} -dU + \frac{i}{2}e^{2U}k &= -\frac{i}{2}\lambda e^{K/2+U}X^I\left(q_I - k\tilde{\zeta}_I - \mathcal{N}_{IJ}(p^J + k\zeta^J)\right) \\ dz^i &= -\frac{i}{2}\lambda e^U g^{i\bar{j}}\bar{f}_{\bar{j}}^I\left(q_I - k\tilde{\zeta}_I - \mathcal{N}_{IJ}(p^J + k\zeta^J)\right) \end{aligned}$$

where λ is fixed by the requirement that dU is real.

- Using standard special geometry formulae this can be rewritten as

$$-dU + \frac{i}{2}e^{2U}k = -\frac{i}{2}\lambda e^U Z\ , \quad dz^i = -i\lambda\frac{|Z|}{Z}e^U g^{i\bar{j}}\partial_{\bar{j}}|Z|$$

$$Z(p,q,k) = e^{K/2}\left[(q_I - 2k\tilde{\zeta}_I)X^I - (p^I + 2k\zeta^I)F_I\right]$$

This generalizes the standard attractor flow equations to non zero NUT charge.

Black holes and D-instantons

- The equivalence between the BH attractor equations and geodesic motion on c-map(M_4) was first observed in the study of spherically symmetric D-instanton solutions in $N = 2$ SUGRA in 5 dimensions: p^I and q_I are M2-brane instanton charge, while k is the M5-brane instanton charge. In fact, such instantons are T-dual to stationary black holes.

  Gutperle and Spalinski; Behrndt Gaida Luest Mahapatra Mohaupt

- This suggests how to incorporate higher-derivative corrections: by mirror symmetry, the $F_h R^2 F^{2h-2}$ corrections in 4D are mapped to

$$\sum_{h=1}^{\infty} \tilde{F}_h \partial^2 S \partial^2 S (\partial C)^{2h-2}$$

  which depend on the hypers only. The reduction to 3D gives rise to higher derivative corrections to the geodesic motion.

  Antoniadis Gava Narain Taylor

- Throughout this lecture, we will omit higher-derivative F-terms.

The universal $SU(2,1)$ sector

- It is instructive to investigate the "universal sector", which encodes the scale U, the graviphoton electric and magnetic charges, and the NUT charge k (this amounts to truncating all moduli away). The Hamiltonian is

$$H = \frac{1}{8}(p_U)^2 - \frac{1}{4}e^{2U}\left[(p_{\tilde{\zeta}} - k\zeta)^2 + (p_\zeta + k\tilde{\zeta})^2\right] + \frac{1}{2}e^{4U}k^2$$

Gauge conditions are $U = \zeta = \tilde{\zeta} = a = 0$ at $\tau = 0$.

- The motion in the $(\tilde{\zeta}, \zeta)$ plane is that of a charged particle in a constant magnetic field. The electric, magnetic charges are the generators of translations; together with the angular momentum

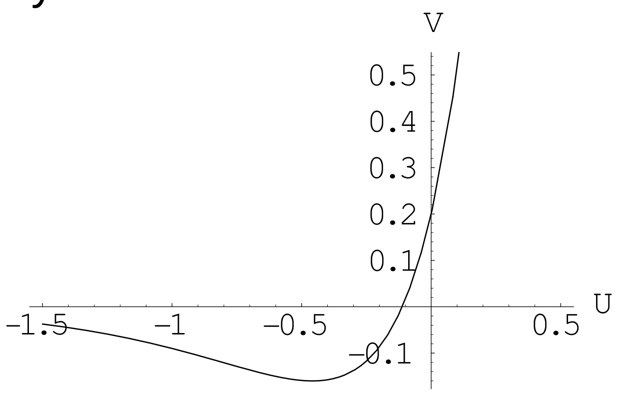
$$p = p_{\tilde{\zeta}} + \zeta k \;, \quad q = p_\zeta - \tilde{\zeta} k \;, \quad J = \zeta p_{\tilde{\zeta}} - \tilde{\zeta} p_\zeta$$

they satisfy the usual magnetic translation algebra

$$[p, q] = k \;, [J, p] = q \;, [J, q] = -p$$

- The motion in the U direction is governed effectively by

$$H = \frac{1}{8}(p_U)^2 + \frac{1}{2}e^{4U}k^2 - \frac{1}{4}e^{2U}\left[p^2 + q^2 - 4kJ\right]$$



- At spatial infinity, p_U becomes equal to the ADM mass, and J vanishes; hence the BPS mass relation

$$M^2 + k^2 = p^2 + q^2$$

- At the horizon $U \to -\infty$, $\tau \to \infty$, the last term is irrelevant and one recovers $AdS_2 \times S_2$ geometry with area

$$A = 2\pi(p^2 + q^2) = 2\pi\sqrt{(p^2 + q^2)^2}$$

- Since V_α^A is a 2×2 matrix, SUSY is equivalent to $H = \det(V_\alpha^A) = 0$:

$$H = \frac{1}{2}\left|p_U + ike^{2U}\right|^2 - \frac{1}{4}e^{2U}\left|p + iq\right|^2 = 0$$

Geodesic motion and nilpotent co-adjoint orbits

- By construction, the Hamiltonian admits a symmetry $G_3 = SU(2,1)$ Positive roots are the standard Heisenberg algebra, negative roots correspond to Ehlers and Harrison transformations.

Kinnersley

- The corresponding Noether charges can be arranged in a matrix Q valued in the (dual) Lie algebra $su(2,1)$, such that

$$H = \text{Tr}(Q^2) \;, \quad \det(Q) = 0$$

The last condition can be checked explicitely, and is necessary in order for the motion not to be over-determined. Different trajectories are related by the co-adjoint action $Q \to hQh^{-1}$ of G on g^*.

- SUSY solutions have $H = 0$. The Cayley-Hamilton theorem for 3x3 matrices implies that $Q^3 = 0$ as a matrix equation (in the fundamental representation).
- In other words, the SUSY phase space is a nilpotent coadjoint orbit of G_3. It inherits a symplectic structure by the standard Kirillov-Kostant method.

$N = 8$ attractors and geodesic motion

- For $N = 8$ SUGRA,

$$\mathcal{M}_3 = E_{8(8)}/SO(16) \ , \quad \mathcal{M}_3^* = E_{8(8)}/SO^*(16)$$

- The SUSY variation is

$$\delta\lambda_A = \epsilon_I \Gamma^I_{A\dot{A}} P^{\dot{A}}$$

  where ϵ_I is a vector of the R-symmetry group in 3 dimensions $SO^*(16)$, $P^{\dot{A}}$ is a 128 spinor of $SO^*(16)$ corresponding to the tangent space to $E_{8(8)}/SO^*(16)$, and λ_A is a conjugate spinor.
- This may be interpreted as a Dirac equation in 16 dimensions, where ϵ_I is the momentum, hence ϵ_I should be light-like. In order to have an ϵ_I such that (*) vanishes, $P^{\dot{A}}$ should be a special spinor.
- For example, 1/2-SUSY trajectories correspond to pure spinors of $SO^*(16)$, of real dimension 58. This is the dimension of the minimal nilpotent orbit of $E_{8(8)}$.

$N = 4$ attractors and geodesic motion

- For $N = 4$ SUGRA with n_v vector multiplets,

$$\mathcal{M}_3 = \frac{SO(8, n_v + 2)}{SO(8) \times SO(n_v + 2)}\,, \quad \mathcal{M}_3^* = \frac{SO(8, n_v + 2)}{SO(6, 2) \times SO(2, n_v)}$$

- The SUSY variation is

$$\delta\lambda^a_A = \epsilon_I \Gamma^I_{A\dot{A}} V^{\dot{A},a}$$

where ϵ_I is a vector R-symmetry group $SO(6, 2)$, and $V^{\dot{A},a}$ ($a = 1 \ldots n_v$), is a collection of n_v spinors of $SO(6, 2)$ corresponding to the tangent space of $SO(8, n_v)/SO(6, 2) \times SO(2, n_v - 2)$.

- SUSY solutions can be obtained by requiring that $V^{\dot{A},a} = \lambda^{\dot{A}} v^a$. 1/2 SUSY trajectories correspond to pure spinors of $SO(6, 2)$, hence the dimension is $n_v + 5$. This is the dimension of the minimal nilpotent orbit of $SO(8, n_v)$.

Very special $N = 2$ supergravity

- Recall that there is an interesting class of $N = 2$ supergravities where the moduli space is a symmetric space. Their prepotential is purely cubic

$$F = N(X)/X^0 = C_{ABC} X^A X^B X^C / X^0$$

where $N(X)$ is the norm of a degree 3 Jordan algebra J. Equivalently, it is invariant under Legendre transform in all variables.

Gunaydin Sierra Townsend

- The 4D moduli space is a symmetric space

$$M_4 = \frac{\text{Conf}(J)}{\text{Lorentz}^c(J) \times U(1)}$$

where $\text{Lorentz}^c(J)$ is the compact form of the reduced structure group of J, while $\text{Conf}(J)$ is the conformal group leaving the cubic light-cone $N(X) = 0$ invariant; Equivalently, it leaves invariant the quartic

$$I_4(p,q) = 4p^0 I_3(q_A) - 4q_0 I_3(p^A) + 4\frac{\partial I_3(q_A)}{\partial q_A}\frac{\partial I_3(p^A)}{\partial p^A} - (p^0 q_0 + p^A q_A)^2$$

Very special $N = 2$ attractors

- Upon compactification to 3 dimensions, the scalar manifold is a symmetric quaternionic-Kahler manifold in Alexseevski's classification:

$$\mathcal{M}_3 = \frac{\text{QConf}(J)}{\text{Conf}^c(J) \times SU(2)}, \quad \mathcal{M}_3^* = \frac{\text{QConf}(J)}{\text{Conf}(J) \times Sl(2)}$$

The 3D U-duality group $G_3 = \text{QConf}(J)$ is the called the quasi-conformal group of J, for reasons to be explained shortly. It contains as subgroups the Heisenberg algebra $[p^I, q_J] = \delta^I_J$ together with the 4D U-duality group $\text{Conf}(J)$, according to the 5-grading

$$\text{QConf}(J) = G_{-2} \oplus G_{-1} \oplus [Conf(J) \times R]_0 \oplus \{p^I, q_I\}_{+1} \oplus \{k\}_{+2}$$

- The SUSY condition is that the Noether charge $Q \in \text{QConf}(J)$ can be conjugated into the grade +1 space. Equivalently,

$$[Ad(Q)]^5 = 0$$

Thus, the SUSY phase space is again a nilpotent coadjoint orbit of the 3D U-duality group.

Q	$D=5$	$D=4$	$D=3$	$D=3^*$
8		$\frac{SU(n,1)}{SU(n)\times U(1)}$	$\frac{SU(n+1,2)}{SU(n+1)\times SU(2)\times U(1)}$	$\frac{SU(n+1,2)}{SU(n,1)\times Sl(2)\times U(1)}$
8	$\mathbb{R}\times\frac{SO(n-1,1)}{SO(n-1)}$	$\frac{SO(n,2)}{SO(n)\times SO(2)}\times\frac{Sl(2)}{U(1)}$	$\frac{SO(n+2,4)}{SO(n+2)\times SO(4)}$	$\frac{SO(n+2,4)}{SO(n,2)\times SO(2,2)}$
8		$\varnothing$	$\frac{SU(2,1)}{SU(2)\times U(1)}$	$\frac{SU(2,1)}{Sl(2)\times U(1)}$
8	$\varnothing$	$\frac{Sl(2)}{U(1)}$	$\frac{G_{2(2)}}{SO(4)}$	$\frac{G_{2(2)}}{SO(2,2)}$
8	$\frac{Sl(3)}{SO(3)}$	$\frac{Sp(6)}{SU(3)\times U(1)}$	$\frac{F_{4(4)}}{USp(6)\times SU(2)}$	$\frac{F_{4(4)}}{Sp(6)\times Sl(2)}$
8	$\frac{Sl(3,C)}{SU(3)}$	$\frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}$	$\frac{E_{6(+2)}}{SU(6)\times SU(2)}$	$\frac{E_{6(+2)}}{SU(3,3)\times Sl(2)}$
24	$\frac{SU^*(6)}{USp(6)}$	$\frac{SO^*(12)}{SU(6)\times U(1)}$	$\frac{E_{7(-5)}}{SO(12)\times SU(2)}$	$\frac{E_{7(-5)}}{SO^*(12)\times Sl(2)}$
8	$\frac{E_{6(-26)}}{F_4}$	$\frac{E_{7(-25)}}{E_6\times U(1)}$	$\frac{E_{8(-24)}}{E_7\times SU(2)}$	$\frac{E_{8(-24)}}{E_{7(-25)}\times Sl(2)}$
10			$\frac{Sp(2n,4)}{Sp(2n)\times Sp(4)}$	?
12			$\frac{SU(n,4)}{SU(n)\times SU(4)}$	?
16	$\mathbb{R}\times\frac{SO(n-5,5)}{SO(n-5)\times SO(5)}$	$\frac{Sl(2)}{U(1)}\times\frac{SO(n-4,6)}{SO(n-4)\times SO(6)}$	$\frac{SO(n-2,8)}{SO(n-2)\times SO(8)}$	$\frac{SO(n-2,8)}{SO(n-4,2)\times SO(2,6)}$
18			$\frac{F_{4(-20)}}{SO(9)}$	?
20		$\frac{SU(5,1)}{SU(5)\times U(1)}$	$\frac{E_{6(-14)}}{SO(10)\times SO(2)}$	$\frac{E_{6(-14)}}{SO^*(10)\times SO(2)}$
32	$\frac{E_{6(6)}}{USp(8)}$	$\frac{E_{7(7)}}{SU(8)}$	$\frac{E_{8(8)}}{SO(16)}$	$\frac{E_{8(8)}}{SO^*(16)}$

The quasiconformal realization

- Due to the above 5-grading, QConf(J) admits a non-linear action on $2n_v+1$ variables $Q=\{p^I, q_I, k\}$. It can be shown that this action leaves the "relative quartic light-cone" invariant:

$$\Delta(Q,Q') = I_4\left(p^I - p'^I, q^I - q'^I\right) + 2\left(k - k' + p'^I q_I - p^I q'_I\right)^2 = 0$$

Gunaydin Koepsell Nicolai; Gunaydin Neitzke BP Waldron

- The physical interpretation of $\Delta(Q,Q')$ is unclear at this moment, but seem to involve bound states of two black holes with relatively non-local charges.
- Moreover, the action of QConf(J) preserves the orbit of (p^I, q_I) under the 4D U-duality group. These orbits are characterized by the number of independent charges:

Ferrara Gunaydin

dim	Constraint on (p,q)	♯charges
$2n_v+1$	$I_4 \neq 0$	4
$2n_v$	$I_4 = 0$	3
$(5n_v-2)/3$	$\partial I_4(p,q) = 0$	2
n_v+2	$\partial \otimes \partial\|_{Conf(J)} I_4(p,q) = 0$	1

The action of QConf(J) on the smallest orbit is in fact the minimal representation of $G_3 = $ QConf(J).

Co-adjoint orbits as phase spaces

- Recall that the Noether charges take values in the dual of the Lie algebra g^*. This is foliated into orbits of the action of G. Each orbit is a symmetric space

$$\mathcal{O}_J = \{g^{-1} J g, g \in G\} = G/Stab(J)$$

where Stab(J) is the stabilizer of J.

- Each orbit carries a natural G-invariant symplectic form, known as the Kirillov-Kostant symplectic form:

$$\omega(X, Y) = \mathrm{Tr}([X, Y]J)$$

on the tangent space around at J. This is evidently non-degenerate (its kernel is given by the commutant of J, which is orthogonal to O_J). Globally,

$$\omega = d\theta \ , \quad \theta = \mathrm{Tr}(g^{-1} dg \ J)$$

where g is a gauge-fixed element in $G/Stab$.

Nilpotent orbits as small phase spaces

- Generic orbits correspond to orbits of semi-simple (=diagonalizable) elements, whose stabilizer is $U(1)^r$, where r is the rank. Their dimension is $\dim G -$ rank$G$ (an even number).
- However, when J has a non-trivial nilpotent part (i.e. non diagonal Jordan form), the stabilizer is typically larger (and non semi-simple), hence the orbit is smaller. Nilpotent orbits are classified by homomorphisms of $Sl(2)$ into G. The smallest orbit is that of a root.
- As an example, the generic orbit of $SU(2,1)$ has dimension 6. The maximal (or regular) nilpotent orbit has the same dimension 6, but the Casimirs are forced to vanish. The minimal (or sub-regular) nilpotent orbit has dimension 4.
- As another example, the generic orbit of $E_{8(8)}$ has dimension 240. The smallest nilpotent orbits have dimension . . . , 114, 112, 92, 58.

The orbit method

- Since the action of G on $\mathcal{O}_J$ preserves the symplectic form, its action on functions on $\mathcal{O}_J$ may be expressed in terms of Poisson brackets. The moment map Q for this symplectic action takes value in the dual of the Lie algebra, in the orbit of J itself.
- The general "orbit method philosophy" indicates that (most of the) unitary representations of G may be obtained by quantizing the Hamiltonian action of G on $\mathcal{O}_J$.
- For example, the regular representation of G on $L^2(G/K)$ at fixed values of the Casimirs (assuming that G is split and K is its maximal compact subgroup) is associated to the orbit of a generic semi-simple element:

$$\dim(G/\text{Stab}) = \dim G - \text{rank}G \,, \qquad \dim(G/K) = (\dim G + \text{rank}G)/2$$

  This is the Hilbert space obtained by quantizing geodesic motion on G/K, at fixed values of the rank$G$ Casimirs !
- Similarly, nilpotent orbits are associated to "unipotent representations" of G, of unusually small dimension.

The quantum attractor mechanism

- The standard way to quantize geodesic motion of a particle on $R^+ \times \mathcal{M}_3^*$ is to replace the classical trajectories by wave functions on $R^+ \times \mathcal{M}_3^*$, satisfying the WdW equation

$$\left[-\frac{\partial^2}{\partial r^2} + \frac{\Delta}{r^2} - 1\right] \Psi(r, U, z^i, \bar{z}^{\bar{i}}, \zeta^I, \tilde{\zeta}_I, a) = 0$$

  where Δ is the Laplace-Beltrami operator on $\mathcal{M}_3^*$.

- As a matter of fact, we have to deal with the geodesic motion of a superparticle, since it comes by reduction from SUGRA in 4D. The wave function is therefore a section of the spinor bundle on $\mathcal{M}_3^*$, or equivalently a set of differential forms on $\mathcal{M}_3^*$.
- Moreover, we are really interested in the SUSY Hilbert space, satisfying the stronger constraint

$$\exists \epsilon / \ \epsilon^\alpha \frac{\partial}{\partial X^A_\alpha} \Psi = 0$$

The BPS Hilbert space

- At fixed (projective) ϵ, this implies that the function does not depend on half of the coordinates X^A. Ψ should be a holomorphic function with respect to the complex structure determined by ϵ^α.
- Better to say, Ψ should be a holomorphic function (or an element of the sheaf cohomology group $H_l(T, O(-h))$ for some l, h) on the twistor space T over the quaternionic-Kahler space $\mathcal{M}_3$. This can be viewed as a higher dimensional, quaternionic version of the Penrose - Atiyah Hitchin Singer twistor tranform.

Salamon; Baston

- More generally, it may be fruitful to consider the hyperkahler cone (HKC) over the quaternionic-Kahler manifold $\mathcal{M}_3$, by including the cone direction r and an extra conjugate variable together with the twistor fiber. The minimal representation of G, relevant for BPS states with 16 supercharges, should then consist of tri-holomorphic functions on HKC.

SUSY Hilbert space for motion on symmetric spaces

- In the case where $\mathcal{M}_3^*$ is a symmetric space G/K, the Hilbert space H may be decomposed into unitary representations $\rho_i : G \longrightarrow H_i$ of G. Furthermore their should exist a map between vectors of each representation and the unconstrained Hilbert space $L^2(G/K)$.
- CAUTION: we are dealing with unitary representations of non-compact groups, hence of infinite dimension. Their size may still be characterized by their Gelfand-Kirillov (or functional) dimension, very roughly, the number d such that $H \sim L_2(R^d)$.
- This can be achieved if the representation admits a (preferably unique) vector f_K, called "spherical vector", invariant under K. Then

$$\Psi(g) = \langle f_K, \rho(g) v \rangle$$

  is K-invariant for any choice of v. If f_K does not exist, any other finite-dim irrep of K (called K-type) will do, and yield a section of some non-trivial bundle over G/H rather than a function.
- Supersymmetric geodesic motion should correspond to unitary representations in a Hilbert space H_{BPS} of unusually small functional dimension: the unipotent representations attached to the nilpotent orbits !

Quaternionic discrete series and very special SUGRA

- Gross and Wallach have constructed unitary representations π_h of G by considering the sheaf cohomology group $H^1(T, O(-h))$ on the twistor space T over the quaternionic-Kahler space $\mathcal{M}_3 = G/K$. For $h \geq 2n_v + 1$, this representation is irreducible, lies in the "quaternionic" discrete series and has functional dimension $2n_v + 1$.
- For lower values of h, the representation becomes decomposable. It admits a unitarizable submodule π'_h of smaller functional dimension:

k	dim	Constraint on (p,q)
$\geq 2n_v + 1$	$2n_v + 1$	$I_4 \neq 0$
$n_v - 1$	$2n_v$	$I_4 = 0$
$(2n_v - 2)/3$	$(5n_v - 2)/3$	$\partial I_4(p,q) = 0$
$(n_v + 2)/3$	$n_v + 2$	$\partial \otimes \partial\|_{Conf(J)} I_4(p,q) = 0$

- These are exactly the quasi-conformal action on (p^I, q_I, k), and its restrictions to the various U-duality orbits !

Quaternionic discrete series and N=4,8 SUGRA

- For example, for $E_{8(-24)}$, the unipotent reps attached to the smallest reps of dim 114,112,92,58 have dimension 57,56,46,29: those are exactly the dimensions of the quasiconformal representations for 4,3,2,1 charge black holes ! Note that all preserve the same amount of SUSY. Optimistically, h may be related to the order of the helicity supertrace...
- After analytic continuation to $E_{8(8)}$, we obtain unipotent reps of dimension 57,56,46,29 corresponding to the BPS Hilbert space of 1/8 BPS, small 1/8 BPS, 1/4 BPS and 1/2 BPS black holes !
- Since the maximal compact group changes, the spherical vector however will be different.
- For $G = E_{8(8)}$ (and all other simply laced groups in their split real form), the minimal representation and its spherical vector have been constructed (although with a totally different motivation). This relies crucially on the invariance of $\exp(I_3(X)/X^0)$ under Fourier. Remarkably,

$$\lim_{\beta\to\infty} e^{\beta H_\omega} f_K = e^{iI_3(\chi^A)/\chi^0}$$

reproduce the tree-level topological amplitude !

Kazhdan Pioline Waldron; BP; Gunaydin Neitzke BP Waldron

Physical interpretation of the wave function

- As usual in diffeomorphism invariant theories (e.g. quantum cosmology), the wave function is independent of the "time" variable ρ, and some other variable should be chosen as a "clock".
- It is natural to use e^U as the "radial clock", since it goes from 0 at the horizon to ∞ at spatial infinity. One could also use the black hole area $A = e^{-2U} r^2$, although classically its range depends on the charges. We expect the wave function to be peaked towards the attractor values of the moduli and the horizon area as $U \to -\infty$.
- The natural inner product is obtained by using the Klein-Gordon inner product (also known as Wronskian, or $U(1)$ charge) at fixed values of U. E.g, the mean value of the horizon area should be roughly

$$A \sim e^{-2U} \int r^2 dr\, dz^i\, d\bar{z}^{\bar{j}}\, \Psi^* \overleftrightarrow{\partial_U}\, \Psi|_{U\to-\infty}$$

- Unfortunately, this product is famously known NOT to be positive definite. A possible way out is "third quantization", where the wave function Ψ becomes itself an operator... this may describe the possible black hole fragmentation near the horizon...

Topological amplitude and spherical vector

- Recall the OSV proposal for BH degeneracies

$$\Omega(p,q) = \langle \Psi_{p,q} | \Psi_{p,q} \rangle \; , \quad \Psi_{p,q}(\chi) = V_{p,q}\Psi_{top} = e^{iq\chi}\Psi_{top}(\chi - p)$$

  interpreted as the overlap between two wave functions associated to each boundary of AdS_2. What is so special about Ψ_{top} ? Do we really need to restrict to $k = 0$?

- On the other hand, we have shown that the proper Hilbert space for the quantum attractor flow is a sub-module $H_{BPS} \subset H \sim L_2(\mathcal{M}_3)$, corresponding to the quantization of BPS geodesic motion on $\mathcal{M}_3$. If $\mathcal{M}_3 = G/K$ is a symmetric space, there is a distinguished "spherical" vector f_K which allows for the map $H_{BPS} \to H$

$$f \longrightarrow \Psi(g) = \langle f, \rho(g) f_K \rangle$$

- We have found circumstancial evidence, at least at tree-level, that (the $k \to 0$ limit of) the spherical vector f_K is in fact the topological string amplitude ! This seems to suggest a 1-parameter extension of the standard topological string amplitude...

The automorphic attractor wave function

- This still leaves an infinite dimensional Hilbert space of BPS wave functions f. A natural physical principle is to select a vector invariant under the 3D U-duality group $G(Z)$:

$$\theta_G(g) = \langle f_{G(\mathbb{Z})}, \rho(g) f_K \rangle$$

is now a function on $G(\mathbb{Z})\backslash G_3(\mathbb{R})/K$, i.e. an automorphic form. This is in fact the general construction of theta series for any group G !

- E.g, the Jacobi theta series

$$\theta(\tau) = \sum_{m \in Z} e^{i\pi m^2 \tau}$$

fits into this frame: τ is an element of $Sl(2)/U(1)$, ρ is the metaplectic representation

$$E_+ = x^2 \, , \quad E_0 = x\partial_x + \partial_x x \, , \quad E_- = \partial_x^2 \, ,$$

f_K is the ground state of the harmonic oscillator, and $f_{G(\mathbb{Z})}$ is the “Dirac comb” distribution $\sum_{m\in\mathbb{Z}} \delta(x-m)$.

BP Waldron Les Houches lecture

Automorphic forms and adeles

- By the "Strong Approximation Theorem", $f_{G(\mathbb{Z})}$ is in fact the product over all primes p of the spherical vector over the p-adic field $\mathbb{Q}_p$. For the Jacobi theta series,

$$\sum_{m\in\mathbb{Z}} \delta(x-m) = \prod_{p\in\mathbb{Z}} \gamma_p(x) \ , \quad \gamma_p(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Z}_p \\ 0 & \text{if} \quad x \notin \mathbb{Z}_p \end{cases}$$

  Indeed, $\gamma_p(x)$ is invariant under p-adic Fourier transform !

- In the language of adeles and ideles,

$$G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R}) = G(\mathbb{Q})\backslash G(\mathbb{A})/K(\mathbb{A})$$

  where $G(\mathbb{Q})$ is diagonally embedded in $G(\mathbb{A})$ and $K(\mathbb{A}) = \prod_p G(\mathbb{Z}_p) \times K(\mathbb{R})$, and the theta series is written adelically as

$$\theta_G(g) = \langle f_{G(\mathbb{Q})}, \rho(g) f_{K(\mathbb{A})} \rangle$$

- The p-adic spherical vector is in fact known for the minimal representation of any simply-laced, split group G.

Kazhdan Polischchuk

Black hole degeneracies and Fourier coefficients

- In the general theory of automorphic forms, Fourier coefficients are associated to choices of parabolic subgroups $P = LN$ of G, and are indexed by characters ξ of P:

$$\hat{\theta}(\xi) = \int_{N(\mathbb{R})/N(\mathbb{Z})} \xi(g)\, \theta_G(g)\, dg$$

- Choosing the maximal (Heisenberg) parabolic subgroup, $N \sim (\zeta^I, \tilde{\zeta}_I, a)$ has two kinds of characters,

$$\xi_{p,q} = e^{i(q_I \zeta^I + p^I \tilde{\zeta}_I)} \quad \text{or} \quad \xi_{p,k} = e^{i(p^I \tilde{\zeta}_I + ka)}$$

In the first case,

$$\hat{\theta}(p,q) = \int d\zeta^I\, d\tilde{\zeta}_I\, da\; e^{i(q_I \zeta^I + p^I \tilde{\zeta}_I)}$$

$$\sum_{(\chi^I, y) \in \mathbb{Q}} \left[e^{i(\tilde{\zeta}_I \chi^I + ay)} f^*_{G(\mathbb{Z})}(\chi^I - \zeta^I, y) \right] \left[e^{i(\tilde{\zeta}_I \chi^I + ay)} f_{K(\mathbb{R})}(\chi^I + \zeta^I, y) \right]$$

Black hole degeneracies and Fourier coefficients (cont)

- The integral of a sets $y = 0$ and the integral over $\tilde{\zeta}_I$ sets $\chi^I = p^I$, hence

$$\hat{\theta}(p, q) = \int d\zeta^I \, e^{iq_I \zeta^I} \, f^*_{G(\mathbb{Z})}(p^I - \zeta^I, 0) \, f_{K(\mathbb{R})}(p^I + \zeta^I, 0)$$

  which is tantalizingly close to the OSV for $\Omega(p, q)$!
- Said otherwise, the automorphic attractor wave function is obtained by choosing the real spherical vector at infinity, and the adelic spherical vector at the horizon. The Fourier coefficients are by construction invariant under $G_4(\mathbb{Z})$.
- It remains to show that $\log \Omega_{p,q} \sim 2\pi \sqrt{I_4(p, q)}$, that the Fourier coefficients are integer, and that they agree with the 4D/5D lift !

Open problems

- Higher derivative corrections
- Rotating and multi-centered black holes in 4D
- Black holes and black rings in 5D
- Automorphic wave functions, and relations to other counting formulae
- Genuine N=2 theories and Kontsevitch's "very wild guess conjecture"
- Time-dependence and midi-superspace models