Black hole degeneracies,
Topological strings
& quantum attractors

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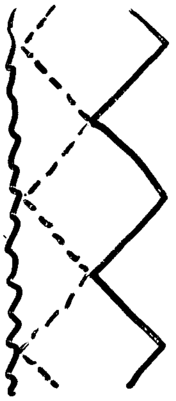
$$S_{\text{thermo}} \overset{?}{\approx} \log \Omega_{\text{micro}}$$
$$\| $$
$$A_H/4G + \dots$$

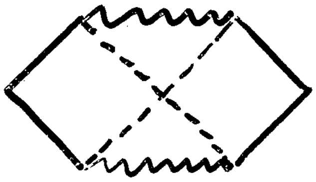
Try going beyond the thermodynamical limit, and understand the microscopic origin of subleading corrections to the Bekenstein - Hawking formula.

Ultimately, compute $\Omega_{\text{micro}}(Q)$ at finite charge.

Alas, not for realistic black holes yet...

Extr. Reissner-Nordström vs Schwarzschild





- time-like singularity
- $T=0$: stable
- near horizon $AdS_2 \times S^2$
- preserves $\frac{1}{2}$ SUSY in $N=2$

- space-like singularity
- $T>0$: evaporates
- near horizon "$Rindler_2 \times S^2$"
- breaks all SUSY

FLOW CHART

$$Y = \begin{cases} \text{CY 3-fold} & \Rightarrow N=2 \\ K3 \times T_2 & \Rightarrow N=4 \\ T^6 & \Rightarrow N=8 \end{cases}$$

$M / Y \times S_1$

IIA / Y $\qquad$ $IIB / \tilde{Y}$

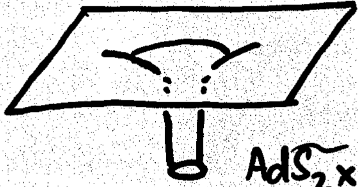
A-model Topological String

B-model Topological String

$D=4, N=2$ SUGRA $+ \sum_{g=0}^{\infty} F_g R^2 T^{2g-2}$

Einstein - Maxwell - Landau

BPS Reissner Nordström BH



Deformed Attractor flow

$$S_{BH_W}(P^I, q_I) = \frac{A_H}{4G} + a_1 \log A/G + b + c\left(\frac{G}{A}\right) + \cdots$$

SPECIFIC THERMODYNAMICAL ENSEMBLE

RADEMACHER CARDY: $\ln \Omega = 2\pi \sqrt{\frac{c}{6} N} + \tilde{a}_1 \log Q + \tilde{b}_1 + \tilde{c}/Q + \cdots$

$D=1+1$ "black string CFT"

D2-D6-NS5-mom $\qquad$ D1-D5-KKM-mom

M2-M5-momentum

Plan of the lectures

[1. Special geometry and attractor flow

2. Very special geometry and black hole entropy in "homogeneous" SUGRA

[3. Basics of topological string theory

[4. Higher derivative corrections and the Ooguri-Verlinde-Vafa conjecture

5. Precision counting of small black holes

[6. The quantum attractor flow

Special geometry

* Recall that N=2 supersymmetry in 4 dimensions allow for two kinds of matter multiplets (besides the gravity multiplet) :

- vector multiplets (A_μ, λ^i, z)

  $z \in \mathcal{M}_V$, a Kähler-Hodge manifold of dimension $2n_V$
  $i=1,2$ R-symmetry index.

- hypermultiplets (Ψ, $\begin{matrix} H \\ \tilde{H}^+ \end{matrix}$, $\tilde{\Psi}$)

  $H, \tilde{H} \in \mathcal{M}_H$, a quaternionic-Kähler space of dimension $4n_H$

At two derivative order, the two interact only gravitationally. The physics of BPS black holes is controlled by the vector multiplets, hypers are spectators.

* The couplings of vector multiplets in N=2 SUGRA are conveniently described by means of an holomorphic section $\Omega = (X^I(z), F_I(z))$ of a $Sp(2n_v+2)$ bundle over $\mathcal{M}_V$, defined up to a scaling $(X^I(z), F_I(z)) \approx (e^{f(z)} X^I(z), e^f F_I(z))$. This section is furthermore constrained to live in a Lagrangian subspace of $\mathbb{C}^{2n_v+2}$, i.e : LAGRANGIAN CONE

$$dX^I \wedge dF_I = 0 \qquad \text{ie} \quad \langle d\Omega, d\Omega \rangle = 0$$

Thus, at generic points one may choose to parametrize $\mathcal{M}_v$ by the "special" homogeneous coordinates $X^I_{I=0..n_v}$, or by the inhomogeneous coordinates X^i/X^0, $i=1..n_v$. The remaining part of the section is then obtained by a "prepotential" $F(X^I)$ such that

$$F_I = \partial F / \partial X^I , \qquad F(e^f X) = e^{2f} F(X)$$

(Note however that this choice obscures symplectic invariance)

* The metric on the manifold $\mathcal{M}_V$ may be obtained from the Kähler potential

$$K(z,\bar z) = -\log\left(i(\bar X^I F_I - X^I \bar F_I)\right) = -\log i\langle\Omega,\bar\Omega\rangle$$

$$g_{i\bar j} = \partial_i\partial_{\bar j} K$$

This is well defined since, under rescalings $X^I \to e^{f(z)}X^I$,

$$K \to K - f(z) - f(\bar z)$$

It will be convenient to define another section $U_i = \overset{\overset{e^{K/2}D_i\Omega}{\shortparallel}}{\left(f_i^I, h_{iI}\right)}$

$$f_i^I = e^{-K/2}\,\partial_i(e^K X^I) = e^{K/2}\left(\partial_i X^I + \partial_i K . X^I\right)$$

$$h_{iI} = e^{-K/2}\,\partial_i(e^K F_I) = e^{K/2}\left(\partial_i F_I + \partial_i K\, F_I\right)$$

(By assumption, $\partial_{\bar i} X^I = \partial_{\bar i} F_I = 0$)

Then the metric may be reexpressed as

$$g_{i\bar j} = -i\langle U_i, U_{\bar j}\rangle = i\left(f_i^I \bar h_{\bar j I} - h_{iI}\bar f_{\bar j}^I\right)$$

while the Riemann tensor of $\mathcal{M}_V$ is

$$(*) \qquad R_{i\bar j k\bar l} = g_{i\bar j}\,g_{k\bar l} + g_{i\bar l}\,g_{k\bar j} - C_{ikm}\,C_{\bar j\bar l\bar m}\,g^{m\bar m}$$

where the tensor C_{ijk} is defined as

$$C_{ijk} = \langle D_i U_j, U_k\rangle = C_{(ijk)}$$

In fact, (*) above is the definition of special geometry.

* The gauge kinetic terms may be written as

$$\mathcal{L} = -\frac{i}{2}\,\bar{\mathcal{N}}_{IJ}\,\mathcal{F}^{-I}_{\mu\nu}\,\mathcal{F}^{-J\mu\nu} + cc$$

where

$$\mathcal{N}_{IJ} = \bar\tau_{IJ} + 2i\,\frac{(\mathrm{Im}\tau.X)_I\,(\mathrm{Im}\tau.X)_J}{X.(\mathrm{Im}\tau)X}$$

In particular, the gauge couplings are $\frac{1}{g^2_{IJ}} = -\mathrm{Im}\,\mathcal{N}_{IJ}$: positive definit

NOTE :

$F_I = \mathcal{N}_{IJ}X^J$

$h_{iI} = \bar{\mathcal{N}}_{IJ}\,f_i^J$

$\tau_{IJ} := \partial_I\partial_J F$

The symplectic symmetries may be made manifest by defining

$$\mathcal{G}_{\mu\nu I} = \bar{\mathcal{N}}_{IJ}\,\mathcal{F}_{\mu\nu}^{J}$$

The conserved electric and magnetic charges are then

$$\begin{pmatrix} p^I \\ q_I \end{pmatrix} = \int_{S_2} \begin{pmatrix} \mathcal{F}^I \\ \mathcal{G}_I \end{pmatrix}$$

The graviphoton is the linear combination

$$T_{\mu\nu} = -2i\left(X.\,\mathrm{Im}\mathcal{N}.\,\mathcal{F}_{\mu\nu}\right).\,e^{K/2}$$

Its associated charge

$$Z = \int_{S_2} T = e^{K/2}\left(q_I X^I - p^I F_I\right)$$

is the central charge of the $N=2$ superalgebra.

BPS states satisfy $\qquad M^2_{BPS} = |Z|^2$

* Electric-magnetic duality acts by symplectic rotations

$$\Omega \to S\,\Omega \qquad \text{where} \qquad S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \begin{aligned} A^tD - C^tB &= \mathbb{1} \\ A^tC &= C^tA \\ B^tD &= D^tB \end{aligned}$$

preserving the symplectic product.

Since $F(X) = \frac{1}{2} X^I F_I$ by homogeneity,

$$\tilde{F}(\tilde{X}) = \frac{1}{2}\left(X^tA^t + F^tB^t\right)\left(CX + DF\right)$$

$$= F(X) + \frac{1}{2} X^t(C^tA)X + \frac{1}{2} F^t(D^tB)F + X^t(C^tB)F$$

where $\tilde{X} = AX + BF$.

This is characteristic of the transformation of a generating function of a Lagrangian submanifold in symplectic geometry.

In particular, for $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\tilde{F}(\tilde{X}) = F(X) - X^t\tilde{X}$ is the Legendre transform of $F(X)$. And in fact, by homogeneity, $\tilde{F}(\tilde{X}) = -F(X)$.

Type IIA compactified on a CY:

$h = h_{1,1}+1$ vector multiplets, coming from reducing the RR 3 forms on $H_2(Y)$. The graviphoton comes from the RR 1-form in 10D.

To leading order,

$$F = -\frac{1}{6} C_{ABC} \frac{X^A X^B X^C}{X^0} \; ; \; K = -\log V$$

The complex scalars are

$$\frac{X^A}{X^0} = \int_{\gamma^A} J \quad , \text{ complexified Kähler structures}$$

$$\frac{F_A}{X^0} = i \int_{\gamma_A} J \wedge J \qquad J = B_{NS} + i\omega_K/\ell_s$$

where C_{ABC} is the intersection form on $H_2 \otimes H_2 \otimes H_2$: $\gamma^A \in H_2$, $\gamma_A \in H_4$

In addition, there are worldsheet instanton corrections known as Gromov-Witten instantons, indexed by rational curves in Y

$$F_{inst} = X_0^2 \sum_{\beta \in H_2(Y,\mathbb{Z})} N_{0,\beta}\, e^{2\pi i X^A/X^0}$$

$$Z = e^{K/2} X^0 \Big(\underset{\boxed{D0}}{q_0} + \underset{\boxed{D2}}{q_A \int_{\gamma_A} J} - \underset{\boxed{D4}}{p^A \int_{\gamma_A} J \wedge J} - \underset{\boxed{D6}}{p^0 \int_{\gamma_A} J \wedge J \wedge J} \Big)$$

Type IIB compactified on CY Y:

$h = h_{2,1}$ vector multiplets, coming from the reduction of the RR 4 form (with self dual 5-form field strength) on $H_3(Y)$.

$$X^I = \int_{A^I} \Omega \quad , \quad F_I = \int_{B_I} \Omega$$

where (A^I, B_I) is a symplectic basis of $H_3(Y)$

$$K = -\log \int \Omega \wedge \bar{\Omega}$$

$$= -\log i(\bar{X}^I F_I - X^I \bar{F}_I) \quad \text{by Riemann's bilinear relations}$$

$$Z = \frac{\int_\gamma \Omega}{\sqrt{\int \Omega \wedge \bar{\Omega}}} \qquad \gamma = q_I A^I - p^I B_I$$

gives the mass of D3 branes wrapped on calibrated (ie special Lagrangian) 3-cycles γ.

Rk: $m_{BPS} = |Z|\, m_p = \left(\frac{V_{CY}}{\ell_s^6}\right)^{1/2} \frac{1}{g_s \ell_s} \cdot e^{K/2} (q_0 X^0 + \dots)$ so indeed $K = -\log(V_Y/\ell_s^6)$ in the gauge $X^0 = 1$

The attractor mechanism

Consider a general ansatz for a static, spherically symmetric black hole in $N=2$ SUGRA:

$$ds^2 = -e^{2U(r)}\, dt^2 + e^{-2U(r)} \left(dr^2 + r^2\, d\Omega_2^2 \right)$$

$$F^{I-} = \frac{1}{2}\left(p^I + i \left(\mathrm{Im}\mathcal{N}_\infty^{-1}\right)^{IJ} \left(\mathrm{Re}\mathcal{N}_{JK}\, p^K - q^J \right)\right) \cdot \left[\sin\theta\, d\theta\, d\varphi - i\frac{e^{2U}}{r^2}\, dt\, dr \right]$$

Assume that the solution preserves one SUSY: then the metric factor $U(r)$ and scalars z are entirely determined by the 1st order equations:

$$\begin{cases} dU/dr = |Z|\, e^{U} / r^2 \\ dz^i/dr = -2\, g^{i\bar{j}}\, e^{U}\, \partial_{\bar{j}} |Z| / r^2 \end{cases}$$

where $Z = e^{K/2}\left(q_I X^I - p^I F_I \right)$ is the central charge.

Defining $\rho = e^{-U}$, the second equation becomes

$$\rho \frac{dz^i}{d\rho} = -g^{i\bar{j}}\, \partial_{\bar{j}} \log |Z|^2$$

which is a gradient flow on $\mathcal{M}_V$ with potential $\log|Z|^2$.

* Near horizon : $r \to 0$

- $dz^i/dr = 0 \quad \Rightarrow \quad \partial_i |Z| = \partial_{\bar{j}} |Z| = 0$
- $e^{-u} du = |Z| dr/r^2 \quad \Rightarrow \quad e^{-u} = \text{cste} + \frac{|Z|}{r}$

The metric becomes

$$ds^2 = -\frac{r^2}{|Z|^2} dt^2 + \frac{|Z|^2}{r^2} \left(dr^2 + r^2 d\Omega_2^2 \right)$$

Define $u = |Z|^2/r$

$$ds^2 = |Z|^2 \left(\frac{-dt^2 + du^2}{u^2} + d\Omega_2^2 \right) \quad : \quad AdS_2 \times S_2$$

The black hole entropy is thus

$$S = \frac{1}{4} . 4\pi |Z|^2 = \pi |Z|^2$$

where Z is the central charge evaluated at the scalars which minimize i

* The attractor equations may be recast as follows:

$$Z = e^{K/2} (q_I X^I - p^I F_I)$$

$$f_i^I = e^{K/2} (\partial_i X^I + \partial_i K \, X^I)$$

hence

$$\partial_i Z = q_I (f_i^I - e^{K/2} \partial_i K \, X^I) - p^I (h_{iI} - e^{K/2} \partial_i K \, F_I) + \frac{1}{2} Z \partial_i K$$

$$\Downarrow$$

$$\partial_i \bar{Z} = \frac{1}{2} \partial_i K \; \bar{Z} \qquad\qquad \partial_i Z = f_i^I (q_I - \bar{\mathcal{N}}_{IJ} p^J) - \frac{1}{2} Z \partial_i K$$

$$\frac{\partial_i |Z|}{|Z|} = \frac{1}{2} \left(\frac{\partial_i Z}{Z} + \frac{\partial_i \bar{Z}}{\bar{Z}} \right) = \frac{1}{2} \frac{f_i^I}{Z} (q_I - \bar{\mathcal{N}}_{IJ} p^J)$$

The attractor point is thus determined by $q_I = \bar{\mathcal{N}}_{IJ} p^J$

or $$q_I = \left(\tau_{IJ} - 2i \frac{(\mathrm{Im}\tau \bar{X})_I (\mathrm{Im}\tau \bar{X})_J}{\bar{X} \, \mathrm{Im}\tau \, \bar{X}} \right) p^J$$

This is rewritten $$q_I - \tau_{IJ} p^J = -i \, C \, (\mathrm{Im}\tau)_{IJ} \bar{X}^J$$

where $$C = +2 \frac{p . \mathrm{Im}\tau . \bar{X}}{\bar{X} . \mathrm{Im}\tau . \bar{X}}$$

Combining this equation with its complex conjugate:

$$q - \tau p = -iC\ (\mathrm{Im}\tau)\ \bar{X}$$
$$q - \bar{\tau} p = +i\bar{C}\ (\mathrm{Im}\tau)\ X$$

$$\Updownarrow$$

$$\begin{array}{l} \mathrm{Re}(CX^I) = p^I \\ \mathrm{Re}(C F_I) = q_I \end{array} \quad \Longleftrightarrow \quad \begin{array}{l} CX^I + \bar{C}\bar{X}^I = 2p^I \\ C\tau_{IJ}X^J + \bar{C}\,\bar{\tau}_{IJ}\,\bar{X}^J = 2q_I \end{array}$$

Using $\mathrm{Im}\mathcal{N} = -\mathrm{Im}\tau + \dfrac{\mathrm{Im}\tau X \otimes \mathrm{Im}\tau X}{X\,\mathrm{Im}\tau X} - \dfrac{\mathrm{Im}\tau\bar{X} \otimes \mathrm{Im}\tau\bar{X}}{\bar{X}\,\mathrm{Im}\tau\bar{X}}$

one easily concludes $\mathrm{Im}\mathcal{N}.X = -\dfrac{(X\,\mathrm{Im}\tau\bar{X})}{\bar{X}\,\mathrm{Im}\tau\bar{X}}\ \mathrm{Im}\tau\,\bar{X}$

hence $C = -2\ \dfrac{p.\,\mathrm{Im}\mathcal{N}.X}{X\,\mathrm{Im}\tau\bar{X}}$

Furthermore $K = -\log(-2\bar{X}\,\mathrm{Im}\tau\bar{X})$

$Z = -2i\ e^{K/2}(p.\mathrm{Im}\mathcal{N}.X)$

hence $C = 2i\,e^{K/2}\,Z$

We may fix the gauge by requiring $|C| = 1$:

$$e^{-K/2} = \frac{|Z|}{2} \sim \frac{r\,e^{-u}}{2} \quad \text{near the horizon}$$

In this gauge, the area of the sphere is related to the Kähler potential via

$$A(r) = 4\pi r^2 e^{-2u} = \pi e^{-K}$$

The phase of C is related to the $N=1$ SUSY preserved ; for single centered black holes, there is no loss of generality in assuming that $C = 1$.

The attractor equations are therefore most easily summarized as

$$\left[\begin{array}{l} \mathrm{Re}(X^I) = p^I \\ \mathrm{Re}(F_I) = q_I \\ \\ \end{array}\right. \qquad S_{BH} = \frac{\pi}{4}\,.\,i(\bar{X}^I F_I - X^I \bar{F}_I)$$

Attractor mechanism and Legendre transform

The first equation $\mathrm{Re}(X^I) = p^I$ may be solved by setting $X^I = p^I + i\varphi^I$
$\bar{X}^I = p^I - i\varphi^I$

The second equation then reads

$$q_I = \frac{1}{2}\left(\frac{\partial F}{\partial X^I} + \frac{\partial \bar{F}}{\partial \bar{X}^I}\right) = \frac{1}{2i}\left(\frac{\partial F}{\partial \phi^I} - \frac{\partial \bar{F}}{\partial \phi^I}\right) = -\frac{1}{\pi}\frac{\partial \mathcal{F}}{\partial \phi^I}$$

Where we define $\mathcal{F}(p,\phi) = \pi \mathrm{Im}\left[F(p+i\varphi)\right]$

The entropy is then

$$\begin{aligned} S_{BH} &= \frac{i\pi}{4}\left[(X^I - 2i\varphi^I)F_I - (\bar{X}^I + 2i\varphi^I)\bar{F}_I\right] \\ &= \frac{i\pi}{2}(F - \bar{F}) + \frac{\pi}{2}\varphi^I(F_I + \bar{F}_I) \\ &= \mathcal{F}(p^I, \varphi^I) + \pi\varphi^I q_I \end{aligned}$$

so that S_{BH} is the Legendre transform of the "free energy" $\mathcal{F}(p^I, \phi^I)$ with respect to the "electric potential" φ^I !

We will see in lect 3 that this continues to hold when higher-derivative F-term interactions are included.

Ex: leading entropy for black holes with $p^0=0$

$$F = -\frac{1}{6} C_{ABC} \frac{X^A X^B X^C}{X^0}$$

$$\mathcal{F} = \frac{\pi}{6} \, \mathrm{Im} \frac{(p^A+i\varphi^A)(p^B+i\varphi^B)(p^C+i\varphi^C)}{p^0+i\varphi^0} \, C_{ABC}$$

$$= -\frac{\pi}{6} \, \mathrm{Re} \frac{(p^A+i\varphi^A)(p^B+i\varphi^B)(p^C+i\varphi^C)}{\varphi^0} \, C_{ABC}$$

$$= -\frac{\pi}{6} \frac{C(p)}{\varphi^0} + \frac{\pi}{2} \frac{C_{AB}(p)\,\varphi^A\varphi^B}{\varphi^0}$$

where $C(p) = C_{ABC}\, p^A p^B p^C$

$C_{AB}(p) = C_{ABC}\, p^C$

$$S_{BH} = \left\langle \mathcal{F} + \pi\left(\varphi^0 q_0 + \varphi^A q_A\right) \right\rangle_{\varphi^0, \varphi^A}$$

The extremization wrt φ^A is Gaussian: $\varphi^A_* = -C^{AB}(p)\, q_B\, \varphi^0$

$$S_{BH} = \left\langle -\frac{\pi}{6}\frac{C(p)}{\varphi^0} + \frac{\pi}{2}\varphi^0 q_A C^{AB}(p) q_B + \pi \varphi^0 q_0 \right\rangle_{\varphi^0}$$

Define $\hat{q}_0 = q_0 + \frac{1}{2} q_A C^{AB}(p) q_B$:

$$S_{BH} = \left\langle -\frac{\pi}{6}\frac{C(p)}{\varphi^0} + \pi \varphi^0 \hat{q}_0 \right\rangle_{\varphi^0}$$

This is stationary at $\varphi^0_* = \pm\sqrt{\frac{-\hat{C}(p)}{6\hat{q}_0}}$ if $\hat{C}(p) > 0$, $\hat{q}_0 < 0$

Choosing the + sign,

$$S_{BH} = 2\pi \sqrt{-\frac{\hat{C}(p)}{6}\,\hat{q}_0}$$

Choosing $q_A = 0$, we get more simply

$$S_{BH} = 2\pi \sqrt{-\, C_{ABC}\, p^A p^B p^C q_0}$$

Very special supergravities

For our purpose to understand black hole entropy in $N>2$ supergravity, it will be useful to consider a class of $N=2$ theories where the moduli space takes a particularly simple form, as a symmetric space G/H. Whether or not these theories exist as true quantum theories is still unknown.

The simplest way of constructing them is to start from 5 dimensions:

$$S = \int d^5x \sqrt{g} \left(R - a_{AB} F^A_{\mu\nu} F^{\mu\nu B} + g_{ij} \partial\phi^i \partial\phi^j \right) + \int C_{ABC} \, A^A \wedge F^B \wedge F^C$$

8 SUSY $\Rightarrow$
- C_{ABC} are constants
- ϕ^i take values in hypersurface $\mathcal{N}=1$, where $\mathcal{N} = C_{ABC} \xi^A \xi^B \xi^C$, with metric g_{ij} induced from $\mathbb{R}^n$

If one furthermore requires that the moduli space is a symmetric space, one finds that $\mathcal{N}(\xi)$ should be the norm of a Jordan algebra of degree 3:
- $j_1 \circ j_2 = j_2 \circ j_1$
- $j_1 \circ (j_2 \circ j_1^2) = (j_1 \circ j_2) \circ j_1^2$

Solutions were classified by Jordan, Von Neuman, Wigner '34:

1. $\mathbb{R}$, $\mathcal{N}(x) = x^3$

2. $\mathbb{R} \oplus \Gamma$, Γ Clifford algebra, $\mathcal{N}(x) = x_1 x^a Q_{ab} x^b$

3. 3×3 hermitean matrices over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ ("division algebras")

$$j = \begin{pmatrix} \alpha_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \alpha_2 & x_1 \\ x_2 & \bar{x}_1 & \alpha_3 \end{pmatrix} \qquad \begin{matrix} \alpha_i \in \mathbb{R} \\ x_i \in \mathbb{K} \end{matrix}$$

$$j_1 \circ j_2 = j_1 \cdot j_2 + j_2 j_1$$

$$* \qquad \mathcal{N}(j) = \alpha_1 \alpha_2 \alpha_3 - \alpha_1 (x_1 \bar{x}_1) - \alpha_2 (x_2 \bar{x}_2) - \alpha_3 (x_3 \bar{x}_3) + 2 \operatorname{Re}(x_1 x_2 x_3)$$

One may then show that the moduli space is

$$\mathcal{M}_J = Str_0(J) / Aut(J)$$

where . $Str_0(J)$ is the "reduced structure group" of J,
ie the group preserving $\mathcal{N}$
. $Aut(J)$ is the automorphism group of J.

*: More explicitly:

J	$\mathcal{N}$	$Str_0(J)$
$J_3^{\mathbb{R}}$	$\det(3 \otimes_s 3)$	$Sl(3, \mathbb{R})$
$J_3^{\mathbb{C}}$	$\det(3 \otimes 3)$	$Sl(3, \mathbb{C})$
$J_3^{\mathbb{H}}$	$Pf(6 \wedge 6)$	$SU^*(6)$
$J_3^{\mathbb{O}}$	$(27 \otimes 27 \otimes 27)\|_s$	$E_{6(-26)}$

Very special geometry and Jordan algebras

N=2	D=5	D=4	D=3	D=3*
$\Gamma_{n-1,1}$	$\frac{SO(n-1,1)\times SO(1,1)}{SO(n-1)}$	$\frac{SO(n,2)}{SO(n)\times SO(2)}\times\frac{Sl(2)}{U(1)}$	$\frac{SO(n+2,4)}{SO(n+2)\times SO(4)}$	$\frac{SO(n+2,4)}{SO(n,2)\times SO(2,2)}$
$\mathbb{R}$	$SO(1,1)$	$\frac{Sl(2)}{U(1)}$	$\frac{G_{2(2)}}{SO(4)}$	$\frac{G_{2(2)}}{SO(2,2)}$
$J_3^{\mathbb{R}}$	$\frac{Sl(3)}{SO(3)}$	$\frac{Sp(6)}{SU(3)\times U(1)}$	$\frac{F_{4(4)}}{USp(6)\times SU(2)}$	$\frac{F_{4(4)}}{Sp(6)\times SU(1,1)}$
$J_3^{\mathbb{C}}$	$\frac{Sl(3,\mathbb{C})}{SU(3)}$	$\frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}$	$\frac{E_{6(2)}}{SU(6)\times SU(2)}$	$\frac{E_{6(2)}}{SU(3,3)\times SU(1,1)}$
$J_3^{\mathbb{H}}$	$\frac{SU^*(6)}{USp(6)}$	$\frac{SO^*(12)}{SU(6)\times U(1)}$	$\frac{E_{7(-5)}}{SO(12)\times SU(2)}$	$\frac{E_{7(-5)}}{SO^*(12)\times SU(1,1}$
$J_3^{\mathbb{O}}$	$\frac{E_{6(-26)}}{F_4}$	$\frac{E_{7(-25)}}{E_6\times U(1)}$	$\frac{E_{8(-24)}}{E_7\times SU(2)}$	$\frac{E_{8(-24)}}{E_{7(-25)}\times SU(1,1}$
N=8				
$J_3^{\mathbb{O}_s}$	$\frac{E_{6(6)}}{USp(8)}$	$\frac{E_{7(7)}}{SU(8)}$	$\frac{E_{8(8)}}{SO(16)}$	$\frac{E_{8(8)}}{SO^*(16)}$
N=4				
$\Gamma_{n-5,5}$	$\frac{SO(n-5,5)}{SO(n-5)\times SO(5)}\times SO(1,1)$	$\frac{SO(n-4,6)}{SO(n-4)\times SO(6)}\times\frac{Sl(2)}{U(1)}$	$\frac{SO(n-2,8)}{SO(n-2)\times SO(8)}$	$\frac{SO(n-2,8)}{SO(n-4,2)\times SO(6,}$

In addition, 4D models which cannot be obtained from 5D:

		$\frac{SU(n,1)}{SU(n)\times U(1)}$	$\frac{SU(n+1,2)}{SU(n+1)\times SU(2)\times U(1)}$	$\frac{SU(n+1,2)}{SU(n,1)\times SU(1,1)\times U}$
$M_{1,2}(O)$		$\frac{E_{6(-14)}}{SO(10)\times U(1)}$	·	·
	↓	↓	↓	↓
	Jordan algebras	Hermitian Jordan triple systems	Freudenthal triple system	(Reduction on a time-like circle)

Reduction 4D → 3D :

$$e_{5D} = \begin{pmatrix} e^{\sigma} & e^{\sigma} B_\mu \\ 0 & e^{-\sigma/2} e_{4D} \end{pmatrix} \qquad \begin{aligned} A_{5D}^{\mathcal{A}} &= (a^A , A_\mu^I) \\ B_\mu &:= A_\mu^0 \end{aligned}$$

Identify $\quad C_{ABC}\, \xi^A \xi^B \xi^C = e^{3\sigma}$

Complexify $\quad t^A = a^A + i \xi^A$

One finds that the scalar fields $t^A, \bar{t}^A$ are inhomogeneous coordinates $t^A = z^A/z^0$ for a special Kähler manifold with prepotential

$$F = \frac{C_{IJK}\, z^I z^J z^K}{z^0}$$

The moduli space is a symmetric space

$$\mathcal{M}_4 = \frac{\mathrm{Conf}(J)}{\widetilde{\mathrm{Str}}_0(J) \times U(1)}$$

where $\mathrm{Conf}(J)$ is the group leaving the "light cone" $\mathcal{N}(\xi) = 0$ invariant, and $\widetilde{\mathrm{Str}}_0(J)$ is its max compact subgroup (in fact, the compact version of $\mathrm{Str}_0(J)$)

Note that $\mathrm{Conf}(J)$ acts linearly on the electric-magnetic charges

$$p^0 , \quad p^A , \quad q_A , \quad q_0$$

Its action preserves the Lagrangian cone $\{ F_I = \partial F / \partial X^I \}$

Therefore, it acts on $\mathcal{M}_4$ by linear fractional transformations.

A remark on Legendre invariance

- An important property following from the "adjoint identity" of Jordan algebras

$$X^{\sharp\sharp} = N(X)X \;, \quad X^{\sharp}_A := C_{ABC}X^B X^C$$

is that F is invariant under Legendre transform in all variables:

$$\langle N(X)/X^0 + X^0 Y_0 + X^A Y_A \rangle_{X^I} = -N(Y)/Y^0$$

Proof: saddle point at $Y_A = X^{\sharp}_A / X^0$, $\quad Y_0 = -N(X)/(X^0)^2$, hence

$$\begin{aligned} N(X)X^A &= (X^0 Y_A)^{\sharp} = (X^0)^2 (Y^A)^{\sharp} \Rightarrow X^A = -Y^{\sharp}_A / Y^0 \\ N(Y)Y_A &= (-X^A Y_0)^{\sharp} \Rightarrow X^0 = N(Y)/(Y_0)^2 \end{aligned}$$

In fact, $(X^0)^{\alpha} N(X)^{\beta} e^{iN(X)/X^0}$ is invariant under Fourier transform, for some choice of α, β !

Etingof Kazhdan Polischchuk

④

BH entropy in very special SUGRA

- As an illustration of the OSV fact, let us compute the tree-level entropy of a black hole with arbitrary charges in very special SUGRA. The free energy is

$$\mathcal{F}(p,\phi) = \frac{\pi}{(p^0)^2+(\phi^0)^2}\left\{p^0\left[\phi^A p^\sharp_A - I_3(\phi)\right] + \phi^0\left[p^A\phi^\sharp_A - I_3(p)\right]\right\}$$

- In order to eliminate the quadratic term in ϕ^A, change variables to

$$x^A = \phi^A - \frac{\phi^0}{p^0}p^A\,, \quad x^0 = [(p^0)^2+(\phi^0)^2]/p^0$$

and, so as to eliminate the square root in $q_0\phi^0$, introduce an auxiliary variable t,

$$\mathcal{S} = \pi\langle -\frac{I_3(x)}{x^0} + \frac{p^\sharp_A + p^0 q_A}{p^0}x^A - \frac{t}{4}\left(\frac{x^0}{p^0}-1\right) - \frac{(2I_3(p)+p^0p^Iq_I)^2}{t\,(p^0)^2}\rangle_{\{x^I,t\}}$$

⑤

BH entropy and 4D/5D lift

Using Legendre invariance, we find

$$S = \pi \left\langle 4 \, \frac{I_3(p^\#_A + p^0 q_A) - [2I_3(p) + p^0 p^I q_I]^2}{(p^0)^2 t} - \frac{t}{4} \right\rangle_t$$

$$= \frac{\pi}{|p^0|} \sqrt{4\, I_3[p^\#_A + p^0 q_A] - [2 I_3(p) + p^0 p^I q_I]^2}$$

This equation has a simple interpretation :

NOTE THAT
5D U-duality $Str_0(J)$
IS MANIFEST

$$S_{4D} = \frac{1}{|p^0|} S_{5D}(Q_A, J)$$

where Q_A, J are the electric charges and angular momentum of a 5D black hole obtained by lifting IIA/CY to M-theory/CY at strong coupling :

$$Q_A = p^0 q_A + C_{ABC}\, p^B p^C$$

$$2J = (p^0)^2 q_0 + p^0 p^A q_A + 2 I_3(p)$$

Conversely, the 4D black hole may be obtained by "compactifying" $M / \mathbb{R}_t \times TN_{p^0} \times CY$ with a 5D black hole at the origin of TN

The TN space has a circle of fixed size at ∞ : translations along this circle correspond to $U(1) \subset SU(2)_L$ rotations around the center.

The charge p^0 TN is obtained by quotienting by $\mathbb{Z}/p^0\mathbb{Z}$: this changes the radius to R/p^0, and the horizon to S^3/p^0 :

$$q_0 = \frac{2J}{(p^0)^2} \quad ; \quad q_A = Q_A/p^0$$

The above formula suggests that this construction works even with $p^A \neq 0$.

$I_3(p) \neq 0$ implies new contributions to the angular momentum J.

Continuing to use the special properties of $I_3(X)$, one may Taylor expand

$$I_3(p_A^\# + p^0 q_A) = I_3(p_A^\#) + p^0 q_A \partial^A I_3(p_A^\#) + (p^0)^2 p_A^\# \partial_A I_3(q_A) + (p^0)^3 I_3(q_A)$$

$$= \left[I_3(p^A)\right]^2 + p^0 I_3(p^A)\, p^A q_A + (p^0)^2 p_A^\# q_\#^A + (p^0)^3 I_3(q_A)$$

This allows to rewrite the entropy as

$$S = \pi \sqrt{I_4(p,q)}$$

where $I_4(p,q) = 4 p^0 I_3(q) - 4 q_0 I_3(p) + 4 p_A^\# q_\#^A - (p^0 q_0 + p^A q_A)^2$

This is the quartic invariant of Conf(J) which follows from Freudenthal's triple system construction

$$\begin{pmatrix} p^0 & p^I \\ q_I & q_0 \end{pmatrix} \qquad p^0, q_0 \in \mathbb{R}, \quad p^I, q_I \in J$$

4D U-duality is thus "manifest".

For $J = J_3^{\mathbb{O}}$:

I_3 = cubic invariant of $E_{6(-26)}$

I_4 = quartic invariant of $E_{7(-25)}$

$J = \mathbb{R} \oplus \Gamma$:

I_3 = cubic inv of $SO(1, n+1)$

I_4 = quartic $Sl(2) \times SO(2, n+2)$

but the same algebra applies for $N=8$ SUGRA :

I_3 = cubic invariant of $E_{6(6)}$

I_4 = quartic invariant of $E_{7(7)}$

$N=4$ SUGRA

$SO(5, n-3)$

$Sl(2) \times SO(6, n-2)$

Basic topological string theory

1. Standard $N=1$ σ-model

$$L = 2t \int d^2z \Big(\frac{1}{2} g_{IJ}(\phi)\, \partial_z \phi^I \partial_{\bar z} \phi^J + \frac{i}{2} g_{IJ}\, \psi_-^I D_z \psi_-^J + \frac{i}{2} g_{IJ}\, \psi_+^I D_{\bar z} \psi_+^J + \frac{1}{4} R_{IJKL}\, \psi_+^I \psi_+^J \psi_-^K \psi_-^L \Big)$$

$\phi : \Sigma \to X$ metric g_{IJ} $\qquad$ TX = complexified tang. bund

ψ_+^I : section of $K^{1/2} \otimes \Phi^*(TX)$ $\qquad$ ε_- : hol section of $K^{-1/2}$

ψ_-^I : $\qquad \bar K^{1/2} \otimes \Phi^*(TX)$ $\qquad$ ε_+ $\qquad \bar K^{-1/2}$

K = canonical bundle on Σ = bundle of $(1,0)$ forms

$\bar K$ = anti " $\qquad\qquad (0,1)$

$D_{\bar z} \psi_+^I = \frac{\partial}{\partial \bar z} \psi_+^I + \frac{\partial \phi^J}{\partial \bar z} \Gamma^I_{JK} \psi_+^K$ is the $\bar\partial$ operator

invariant under

$$\delta \Phi^I = i(\varepsilon_- \psi_+^I + \varepsilon_+ \psi_-^I)$$
$$\delta \psi_+^I = -\varepsilon_- \partial_z \phi^I - i \varepsilon_+ \psi_-^K \Gamma^I_{KM} \psi_+^M$$
$$\delta \psi_-^I = -\varepsilon_+ \partial_{\bar z} \phi^I - i \varepsilon_- \psi_+^K \Gamma^I_{KM} \psi_-^M$$

. yields $N=1$ SCFT : $\quad T, \quad G$

$$T(z)T(w) = \frac{c}{2} \frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots$$
$$T(z)G(w) = \frac{3}{2} \frac{G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w} + \dots$$
$$G(z)G(w) = \frac{2c/3}{(z-w)^3} + 2\frac{T(w)}{z-w} + \dots$$

In the Ramond sector, G has a zero mode G_0 :

$$\{G_0, G_0\} = 2\left(L_0 - \frac{c}{24}\right) \quad \Rightarrow \quad \Delta \geq c/24$$

Witten 9306122

2. N=2 σ model

Assume X is Kähler and choose local complex coordinates $\phi^i, \phi^{\bar i}$

Decompose $TX = T^{1,0}X \oplus T^{0,1}X$

$$L = 2t\int d^2z\left(\frac{1}{2}g_{IJ}\,\partial_z\phi^I\partial_z\phi^J + i\,\psi_-^{\bar i}D_z\psi_-^i\,g_{i\bar i} + i\,\psi_+^{\bar i}D_{\bar z}\psi^i\,g_{i\bar i} + R_{i\bar i j\bar j}\,\psi_+^i\psi_+^{\bar i}\psi_-^j\psi_-^{\bar j}\right)$$

invariant under
$$\delta\phi^i = i(\alpha_-\psi_+^i + a_+\psi_-^i)$$
$$\delta\phi^{\bar i} = i(\check\alpha_-\psi_+^{\bar i} + \tilde\alpha_+\psi^{\bar i})$$

where $\alpha_-, \check\alpha_-$ are sections of $K^{-1/2}$

$\alpha_+, \tilde\alpha_+$ $\bar K^{-1/2}$

This realizes the N=2 SCFT: $T,\ G^\pm,\ J$

$\Delta = 2,\ 3/2,\ 1$

$Q = 0,\ \pm 1,\ 0$

$$G^+G^+ = \text{reg} = G^-G^-$$
$$G^+(z)G^-(w) = \frac{2c}{3}\frac{1}{(z-w)^3} + \left(\frac{2J}{(z-w)^2} + \frac{\partial J}{z-w}\right) + \frac{2T}{z-w} + \dots$$
$$J(z)J(w) = \frac{c/3}{(z-w)^2} + \dots$$

In the Ramond sector,
$$(G_0^+)^2 = (G_0^-)^2 = 0$$
$$\{G_0^+, G_0^-\} = 2\left(L_0^{R\pm} - \frac{c}{24}\right) = 2\left(L_0^{NS} - \frac{1}{2}J_0^{NS}\right) \text{ by } (*)$$

The spectral flow relates NS and R sectors: ⟺ spacetime susy
$$J_0^{R\pm} = J_0^{NS} \mp \frac{c}{6}$$
$$L_0^{R\pm} = L_0^{NS} - \frac{1}{2}J_0^{NS} + \frac{c}{24} \qquad (*)$$

Hence $L_0^{R\pm} - \frac{c}{24} \geqslant 0$ implies $L_0^{NS} - \frac{1}{2}J_0^{NS} \geqslant 0$: $\Delta \geqslant |q|/2$

Such chiral states form the <u>chiral ring</u>

Iterating the spectral flow, we find a map NS → NS

$$J_0' = J_0 \mp \frac{c}{3}$$
$$L_0' = L_0 - J_0 + \frac{c}{6}$$

In particular, the NS ground state maps to a state with $(h,q) = \left(\frac{c}{6}, \mp\frac{c}{3}\right)$

This is in fact the NS state of maximal charge $|q|$ in the chiral ring.

For a Calabi-Yau 3-fold, $c = 9$ so we have

$$(0,0)_{NS} \; ; \; \underbrace{\left(3/8 , \pm 3/2\right)_R}_{\substack{\text{covariantly cst} \\ \text{spinor fields}}} \; ; \; \underbrace{\left(\tfrac{3}{2} \; ; \; \pm 3\right)_{NS}}_{\text{associated to (3,0) form on CY}}$$

The spectral flow above is in fact known as a topological twist: the new spin in the R± sector is $L_0^{NS} - \frac{1}{2} J_0^{NS}$

In particular ψ_+^i now transform as $\Phi^*(T^{1,0}X)$

$\psi_+^{\bar i}$ $\quad K \otimes \Phi^*(T^{0,1}X)$

If we perform the same twist on the right movers,

ψ_-^i transform as $\bar K \otimes \Phi^*(T^{1,0}X)$

$\psi_-^{\bar i}$ $\quad \Phi^*(T^{0,1}X)$

This is known as the A-twist.

If we perform a different twist, e.g

$\psi_+^i \quad K \otimes T^{1,0}$

$\psi_+^{\bar i} \quad T^{0,1}$

$\psi_-^i \quad \bar K \otimes T^{1,0}$

$\psi_-^{\bar i} \quad T^{0,1}$

they agree with the origin model provided $K^{1/2}$ and $\bar K^{1/2}$ are trivial.

we obtain the B-twist.

These two twists correspond to different topological versions of the $N=2$ σ model

Topological A-model

(Marino 0406005)

Here ψ^i_+ and $\psi^{\bar i}_-$ are $\overset{\text{scalars}}{\downarrow}$ in $T^{1,0}X \oplus T^{0,1}X$, combined into $\chi \in TX$

$\psi^{\bar i}_+$ is a $(1,0)$ form : $\psi^{\bar i}_z$

ψ^i_- is a $(0,1)$ form : $\psi^i_{\bar z}$

The lagrangian becomes

$$L = 2t\int d^2z \Big(\tfrac{1}{2} g_{IJ}\,\partial_z\phi^I \partial_{\bar z}\phi^J + i\,\psi^{\bar i}_z D_{\bar z}\chi^i\, g_{i\bar i} + i\,\psi^i_{\bar z} D_z \chi^{\bar i}\, g_{i\bar i} - R_{i\bar i j\bar j}\,\psi^i_{\bar z}\psi^{\bar i}_z \chi^j \chi^{\bar j} \Big)$$

Invariant under

$$\{Q, \phi^I\} = \chi^I$$

$$\{Q, \chi^I\} = 0$$

$$\{Q, \psi^i_{\bar z}\} = i\partial_{\bar z}\phi^i - \chi^j \Gamma^i_{jk}\psi^k_{\bar z}$$

$$\{Q, \psi^{\bar i}_z\} = i\partial_z\phi^{\bar i} - \chi^{\bar j}\Gamma^{\bar i}_{\bar j\bar k}\psi^{\bar k}_z$$

$Q^2 = 0$; ghost number ϕ, χ, ψ, Q is classically conserved
$0, 1, -1, 1$

The lagrangien can be written as

$$\mathcal{L} = -i\{Q, V\} - t\int_{\Sigma_g} \Phi^*(J)$$

Where $\Phi^*(J) = i\, g_{i\bar j}\, d\phi^i \wedge d\phi^{\bar j}$ is the pull back of the Kähler class

$$V = t\int_{\Sigma_g} d^2z \; g_{i\bar j}\left(\psi^i_{\bar z}\partial_z\phi^{\bar j} + \partial_{\bar z}\phi^i \psi^{\bar j}\right)$$

is the "gauge fermion"

It is natural to complexify the Kähler class by adding $-it\int\Phi^*(B)$

Remark that

- the energy momentum tensor on the world sheet is Q-exact

$$T_{\alpha\beta} = \{Q, b_{\alpha\beta}\} \qquad b_{\alpha\beta} = \frac{\delta V}{\delta g^{\alpha\beta}}, \qquad \text{ghost\#} = -1$$

- Aside from the topological term $+it\int \phi^*(iJ+B)$, the partition function is independant of t, and of the cx structure of X. In particular, the semi-classical approximation around $t \to \infty$ is exact: classical solutions are holomorphic maps

$$\partial_{\bar z}\phi^i = 0 \qquad\qquad \text{the fixed points of } Q\text{, really}$$

$$\partial_z \phi^{\bar i} = 0$$

- BRST symmetry ensures that the ratio of bos and ferm determinants is ± 1

The observables of the A-model are in 1-1 correspondance with the de Rham cohomology of X:

$$O_W = W_{I_1 \dots I_n}\chi^{I_1} \cdots \chi^{I_n} \qquad \text{ghost\#} = n$$

Indeed, $\{Q, Q_W\} = -O_{dW}$

Due to an anomaly in the $u(1)$ ghost current, correlators vanish unless

$$\sum_{k=1}^{\ell} \deg(\phi_k) = 2d(1-g) + 2\underbrace{\int_\Sigma \phi^*(c_1(X))}_{0 \text{ for CY}}$$

Eg: for CY3 folds,

- $g=0$: the only non-zero correlator is

$$\langle O_{W_1} O_{W_2} O_{W_3}\rangle = \int W_1 \wedge W_2 \wedge W_3 + \sum_{\beta = \sum n_i [S_i]} \int_\beta W_1 \int_\beta W_2 \int_\beta W_3 \, e^{-\sum t_i n_i}$$

- $g=1$: partition function is the elliptic genus

Topological B-model (requires $c_1(X)=0$ for anomaly cancellation)

Marino 0410165

This time, $\psi_+^{\bar{i}}$ and $\psi_-^{\bar{i}}$ are scalars in $T^{0,1}$

while ψ_+^{i} and ψ_-^{i} are sections of $\underbrace{(K \oplus \bar{K})}_{\text{1-form } \rho} \otimes T^{1,0}$

Define
$$\eta^{\bar{i}} = \psi_+^{\bar{i}} + \psi_-^{\bar{i}}$$
$$\theta_i = g_{i\bar{i}} (\psi_+^{\bar{i}} - \psi_-^{\bar{i}})$$

The Lagrangian is now

$$L = t \int d^2z \Big(g_{IJ} \partial_z \phi^I \partial_{\bar{z}} \phi^J + i \eta^{\bar{i}} (D_z \rho_{\bar{z}}^{i} + D_{\bar{z}} \rho_z^{i}) g_{i\bar{i}}$$
$$+ i\theta_i (D_{\bar{z}} \rho_z^{i} - D_z \rho_{\bar{z}}^{i})$$
$$+ R_{i\bar{i}j\bar{j}} \rho_z^{i} \rho_{\bar{z}}^{j} \eta^{\bar{i}} \theta_k g^{k\bar{j}} \Big)$$
$$= it \int \{Q, V\} + tW$$

Where the gauge fermion $V = g_{i\bar{j}} (\rho_{\bar{z}}^{i} \partial_z \phi^{\bar{j}} + \rho_z^{i} \partial_{\bar{z}} \phi^{\bar{i}})$

and
$$W = \int -\theta_i \underset{\substack{\uparrow \\ \text{exterior derivative}}}{D} \rho^i - \frac{i}{2} R_{i\bar{i}j\bar{j}} \rho^i \wedge \rho^j \eta^{\bar{i}} \theta_k g^{k\bar{j}}$$

not obvious b
↑ tru

Thus B is independant of the cx structure of Σ and of the Kähler metric on X. It does however depend on the cx structure of X

$$\{Q, \phi^i\} = 0$$
$$\{Q, \phi^{\bar{i}}\} = \eta^{\bar{i}}$$
$$\{Q, \eta^{\bar{i}}\} = 0$$
$$\{Q, \theta_i\} = 0$$
$$\{Q, \rho^i\} = i\, d\phi^i$$

ghost #:	η	θ	ρ	ϕ	Q
	1	1	-1	0	1

Again, the semi-classical approximation is exact: the t dependance of the tW term can be reabsorbed by $\theta \to \theta/t$, so the only t-dependance comes as an overall factor in correlation functions.

The classical solutions (fixed points of Q) are constant maps

$$d\phi^i = 0 \;, \quad \eta^{\bar i} = 0$$

$$\eta^{\bar i} \Leftrightarrow \bar\partial$$

Observables: 1-1 correspondence with $\bigoplus_{p,q} H^p(X, \Lambda^q T^{1,0}X)$

$$V = V_{\bar i_1 \dots \bar i_p}{}^{j_1 \dots j_q} \, d\bar z^{i_1} \dots d\bar z^{i_p} \frac{\partial}{\partial z_{j_1}} \dots \frac{\partial}{\partial z_{j_q}}$$

Indeed, $\{Q, \mathcal{O}_V\} = -\mathcal{O}_{\bar\partial V}$

so the BRST cohomology of Q consists of V such that $\bar\partial V = 0$
$V \simeq V + \bar\partial S$

Again, the anomalous ghost number conservation requires

$$\sum p_k = d(1-g)$$
$$\sum q_k = d(1-g)$$

The path integral reduces to an integral over X. For example, for CY 3-folds, $p=q=1$,

$H^1(X, T^{1,0}X) \approx H^{2,1}$: parametrizes complex structures deformations

$$W^i{}_{\bar j} \to \underbrace{\Omega_{ikl}}_{\text{holomorphic 3-form}} W^i{}_{\bar j} \in H^{2,1}$$

$$\langle \mathcal{O}_{W_1} \mathcal{O}_{W_2} \mathcal{O}_{W_3} \rangle = \int_X W_1{}^{i_1}{}_{\bar j_1} W_2{}^{i_2}{}_{\bar j_2} W_3{}^{i_3}{}_{\bar j_3} \, \Omega_{i_1 i_2 i_3} \, d\bar z^{\bar j_1} d\bar z^{\bar j_2} d\bar z^{\bar j_3} \wedge \Omega$$

Special geometry gives this as $\dfrac{\partial^3 F_0}{\partial t_a \partial t_b \partial t_c}$

Coupling to topological gravity

Due to the issue of ghost number conservation, we have seen that only $\langle \mathcal{O}\,\mathcal{O}\,\mathcal{O}\rangle_{\text{sphere}}$ and $\langle 1 \rangle_{\text{torus}}$ are non vanishing.

In order to compute higher genus correlators, it is necessary to couple the A or B topological matter to topological gravity. The answer is roughly as follows:

Recall that in bosonic string, genus g amplitudes are obtained by inserting the "antighost" b of reparametrization invariance, "folded" with Beltrami differentials:

$$F_g = \int_{\overline{\mathcal{M}}_g} \langle \prod_{k=1}^{6g-6} (b, \mu_k) \rangle \qquad \mu_k \in H^1(\Sigma_g, T^1\Sigma_g)$$

where $\overline{\mathcal{M}}_g$ is the "Deligne-Mumford" compactification of moduli space of complex structures of Σ_g, and

$$(b, \mu_k) = \int_{\Sigma_g} d^2z \left[b_{zz} (\mu_k)_{\bar{z}}{}^{z} + b_{\bar{z}\bar{z}} (\bar{\mu}_k)_{z}{}^{\bar{z}} \right]$$

Since b has ghost number -1, this exactly compensates the background charge.

In topological string, G_- has spin 2 so can be identified with b:

$$\begin{aligned} &\text{bosonic string: } T_{\alpha\beta} = \{Q, b_{\alpha\beta}\} \\ &\text{top string: } \quad T = \{G_+, G_-\} \end{aligned} \quad ; \quad F_g^{A/B} = \int_{\mathcal{M}_g} \prod_{a=1}^{3g-3} (\mu_a G_-)(\tilde{\mu}_a \tilde{G}_\pm)$$

"Scattering" amplitudes at higher genus may be defined by including integrated vertex operators:

$$d\mathcal{O}^{(0)} = \{Q, \mathcal{O}^{(1)}\}$$
$$d\mathcal{O}^{(1)} = \{Q, \mathcal{O}^{(2)}\}$$

then $\mathcal{O}^{(2)}$ has ghost number 0 if $\mathcal{O}^{(0)}$ has # 2.

The target space theories are known as
| A: holomorphic Chern Simons
| B: Kodaira-Spencer

Gromov-Witten invariants & Gopakumar Vafa invariants

With this prescription for coupling to topological gravity, the vacuum amplitudes are now non trivial at arbitrary genus.

It is convenient to define a generating function a "string amplitude"

$$F_{top} = \sum_{g=0}^{+\infty} \lambda^{2g-2} F_g$$

where λ is known as the "topological string coupling".

→ t = fixed, $\bar{t} \to \infty$

For the A model on a CY three fold X, the (holomorphic) string amplitude has an asymptotic expansion

$$F_{top} = \overset{\text{genus } 0}{-i \frac{(2\pi)^3}{6\lambda^2} C_{ABC} t^A t^B t^C} \overset{\text{genus } 1}{- \frac{i\pi}{12} c_{2A} t^A} + F_{GW} \qquad (*)$$

where t^A are the complexified Kähler moduli, on a basis γ^A of $H_{1,1}$

C_{ABC} are the triple intersection numbers of the dual 4-cycles

$c_{2A} = \int J_A c_2(T^{1,0}X)$ the second Chern classes

$C_{ABC} = \int J_A \wedge J_B \wedge J_C$

The two first terms in (*) are $\overset{\text{in } \alpha'}{\text{perturbative}}$, while F_{GW} contains the effect of worldsheet instantons at arbitrary genus:

β = effective curve, $\beta_A^A \geqslant 0$

$$F_{GW} = \sum_{g \geqslant 0 ; \beta} N_{g,\beta} \, e^{2\pi i \beta_A t^A} \lambda^{2g-2}$$

$$= \sum_{g \geqslant 0 ; \beta ; d \geqslant 1} n_\beta^g \frac{1}{d} \left[2 \sin\left(\frac{d\lambda}{2}\right) \right]^{2g-2} e^{2\pi i d \beta_A t^A}$$

$N_{g,\beta}$ are rational-valued topological invariants of X known as Gromov-Witten

n_β^g are the Gopakumar-Vafa invariants, believed to be integer: it counts the number of genus g holomorphic curves in the homology class $\beta_A \gamma^A$.

Note in particular that the leading term as $\lambda \to 0$ is

$$\frac{1}{\lambda^2} \sum_\beta N_{g,\beta}\, e^{2\pi i \beta_A t^A} = \sum_\beta n^0_\beta \sum_{d=1}^{\infty} \frac{1}{d^3 \lambda^2}\, e^{2\pi i d \beta_A t^A}$$

which incorporates the effects of multiple coverings for an isolated genus 0 curve.

Note also that for $\beta = 0$, the contributions are no longer exp. suppressed. They occur at genus 0 only, with $n^0_0 = -\frac{1}{2}\chi(X)$

They are given by the Mac-Mahon function

$$f(\lambda) = \sum_{d=1}^{\infty} \frac{1}{d} \frac{1}{\left(2\sin \frac{d\lambda}{2}\right)^2}$$

$$= \sum_{d=1}^{\infty} \frac{1}{d} \left(\frac{i}{e^{\frac{id\lambda}{2}} - e^{-\frac{id\lambda}{2}}} \right)^2 = -\sum_{d=1}^{\infty} \frac{e^{id\lambda}}{d(1-e^{id\lambda})^2}$$

$$= \sum_{d=1}^{\infty} \sum_{n=1}^{\infty} \frac{n q^{nd}}{d} \qquad \text{where } q = e^{i\lambda}$$

$$= + \sum_n \log(1-q^n)$$

The latter form converges in the upper half plane $\mathrm{Im}\,\lambda > 0$, and is suitable to study the large coupling limit $\lambda \to i\infty$.

In the weak coupling limit, the leading term is easy to extract,

$$f(\lambda) \sim \frac{1}{\lambda^2} \sum_{d=1}^{\infty} \frac{1}{d^3} = \zeta(3)/\lambda^2 + \dots$$

This constant shift $-\frac{1}{2}\chi\,\zeta(3)$ of the tree level prepotential can be traced to the $\zeta(3) R^4$ tree-level term in the 10-dimensional action.

The subleading terms are more tricky to extract, but interesting too:

The Mac-Mahon function

$$f(\lambda) = \sum_{n=1}^{\infty} n \log(1-q^n) \qquad q = e^{in\lambda} = e^{-nt}$$

$$= - \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{n\, e^{-nkt}}{k}$$

In order to analyze the behavior at $t \to 0$, we first consider its Mellin transform

$$Mf(s) = \int_0^{+\infty} \frac{dt}{t^{1-s}} f(t)$$

$$= - \sum \frac{n}{k} (\; nk)^{-s} \Gamma(s)$$

$$= - \zeta(s-1) \; \zeta(s+1) \; \Gamma(s)$$

The original function may be obtained conversely by

$$f(t) = \frac{1}{2\pi i} \int_{\mathrm{Re}\, s = \sigma} Mf(s) \; t^{-s}$$

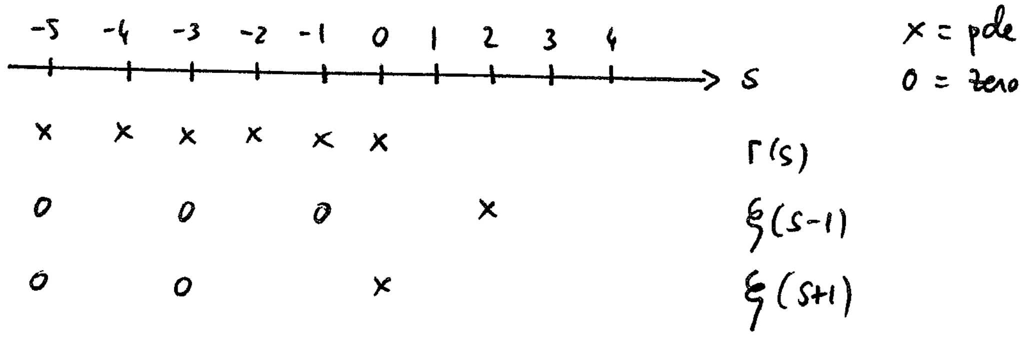
where σ is chosen to lie to the right of any pole of $Mf(s)$. Moving the contour to the left and crossing the poles generate the Laurent series expansion of $f(t)$.

Recall that $\Gamma(s)$ has poles at $s = -n$ $\quad n = 0, 1, \cdots$, residue $\frac{(-1)^n}{n!}$

$\zeta(s)$ has a pole at $s = 1$, $\quad \zeta(s) \sim \frac{1}{s-1}$

and "trivial" zeros at $s = -2, -4, -6, \ldots$

Thus :



$$f(t) = \sum_{g=0}^{+\infty} \underset{s=2-2g}{\text{Residue}} \left[- \zeta(s-1) \; \zeta(s+1) \; \Gamma(s) \; t^{-s} \right]$$

At $s=2$: Residue $= -\zeta(3)/_{t^2}$

$s=0$: double pole ; residue $= -\zeta'(-1) - \frac{1}{12}\log t$

$s=2-2g$, $g\geq 2$: residue $= -\dfrac{\zeta(1-2g)\,\zeta(3-2g)}{(2g-2)!}\, t^{2g-2}$

use $\zeta(3-2g) = -\dfrac{B_{2g-2}}{2g-2}$ $(g\geq 2)$

to obtain the Laurent series expansion

$$f(t) = \frac{\zeta(3)}{\lambda^2} + \frac{1}{12}\log(i/\lambda) - \zeta'(-1) + \sum_{g=2}^{\infty} \frac{\lambda^{2g-2}}{(2g-2)!}\,\frac{B_{2g-2}}{2g-2}\cdot\frac{B_{2g}}{2g}$$

Impressively, this agrees with

$$\int_{\mathcal{M}_g} c_{g-1}^3 = \frac{-B_{2g}\;B_{2g-2}}{2g\,(2g-2).(2g-2)!}$$ Faber - Pandharipande '98

Amusingly, the MacMahon function is also the partition function of three dimensional Young tableaux. This is the simplest occurrence of a general phenomenon which will not be covered here.

Holomorphic anomalies

Naively, a variation of the anti-holomorphic moduli $\bar{t}^{i}$ corresponds to the insertion of a BRST trivial operator

$$\bar{\phi}_{\bar{i}} = \{ G^{+}, [\bar{G}^{+}, \bar{\phi}_{\bar{i}}] \}$$

while a variation of the holomorphic moduli t^{i} would correspond to n-point functions:

$$C^{g}_{i_1 \cdots i_n} = \partial^{n}_{i_1 \cdots i_n} F_g$$

This however makes no sense, since F_g is a non-trivial section of $\mathcal{L}^{2-2g}$. Both of these statements are corrected by taking into account boundary contributions in the integral over moduli of $\mathcal{M}_g$:

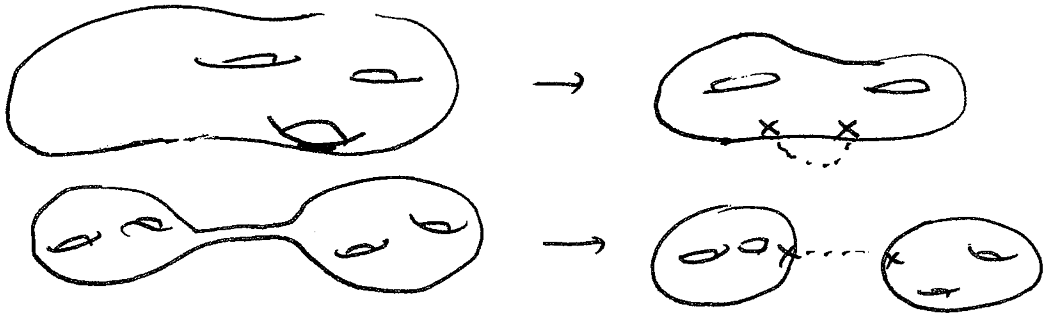
$$\frac{\partial}{\partial \bar{t}^{i}} F_g = \int_{\mathcal{M}_g} \int d^2 z \langle \oint_{C_z} G^{+} \oint_{C'_z} \bar{G}^{+} \bar{\Phi}_{\bar{i}}(z) \prod_{a=1}^{3g-3} \int \mu_a G^{-} \prod_{\bar{a}=1}^{3g-3} \int \bar{\mu}_a \bar{G}^{-} \rangle$$

$$= \sum_{b,\bar{b}=1}^{3g-3} \int_{\mathcal{M}_g} \int \bar{\Phi}_{\bar{i}} \int 2\mu_b T \int 2\bar{\mu}_{\bar{b}} \bar{T} \prod_{\substack{a=1 \\ a\neq b}}^{3g-3} \left(\int \mu_a G^{-} \right) \prod_{\substack{\bar{a}=1 \\ \bar{a}\neq \bar{b}}}^{3g-3} \bar{\mu}_a \bar{G}^{-} \rangle$$

$$= \sum_{b,\bar{b}=1}^{3g-3} \int_{\mathcal{M}_g} \frac{\partial}{\partial m_b} \frac{\partial}{\partial \bar{m}_{\bar{b}}} \langle \int \bar{\Phi}_{\bar{i}} \prod_{a\neq b} \left(\int \mu_a G^{-} \right) \prod_{\bar{a}\neq\bar{b}} \left(\int \mu_{\bar{a}} \bar{G}^{-} \right) \rangle$$

where we used $G^{+}(z) G^{-}(w) = \frac{2T}{z-w}$

There are 2 types of boundaries:



$$\boxed{\bar{\partial}_{\bar{i}} F_g = \frac{1}{2} \bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \left(D_j D_k F_{g-1} + \sum_{r=1}^{g-1} (D_j F_r)(D_k F_{g-r}) \right)}$$

where $D_i = \partial_i - \Gamma_i - (2-2g)\partial_i K$ $\qquad g \geq 2$

There

At genus 1, the holomorphic equation becomes 2nd order:

$$\partial_i \bar{\partial}_{\bar{j}} F_1 = \frac{1}{2} C_{ik\ell} C_{\bar{j}\bar{k}\bar{\ell}} e^{2K} G^{k\bar{k}} G^{\ell\bar{\ell}} - \left(\frac{\chi}{24} - 1\right) G_{i\bar{j}}$$

Similarly, one may derive anomaly equations for correlation functions.

This is best summarized by defining the generating function

$$W(\lambda, x; t, \bar{t}) = \sum_{g=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^{2g-2+n} C^{(g)}_{i_1 \dots i_n} x^{i_1} \dots x^{i_n}$$

$$+ \left(\frac{\chi}{24} - 1\right) \log \lambda$$

differs from BCOV by $x \to \lambda x$
$\frac{\partial}{\partial x} \to \frac{1}{\lambda}\frac{\partial}{\partial x}$, $\frac{\partial}{\partial \lambda} \to \frac{\partial}{\partial \lambda} - \frac{x}{\lambda}\frac{\partial}{\partial x}$

where
$$C^{(0)}_{i_1 \dots i_n} = D_{i_1} \dots D_{i_{n-3}} C_{i_{n-2} i_{n-1} i_n} \quad , \quad n \geqslant 3$$
$$C^{(g)}_{i_1 \dots i_n} = D_{i_1} \dots D_{i_n} F^{(g)} \qquad g \geqslant 1$$
$$= 0 \text{ if } 2g - 2 + n \leq 0$$

so
$$W = \frac{1}{3!} C^{(0)}_{ijk} x^i x^j x^k + \frac{1}{4!} C^{(0)}_{ijk\ell} x^i x^j x^k x^\ell$$
$$+ \left(\lambda C^{(1)}_i x^i + \frac{\lambda^2}{2} C^{(1)}_{ij} x^i x^j + \cdots\right)$$
$$+ \left(\lambda^2 F_2 + \lambda^3 C^{(2)}_i x^i + \cdots\right)$$
$$+ \left(\lambda^4 F_3 + \cdots\right) \qquad + \left(\frac{\chi}{24} - 1\right) \log \lambda$$

This satisfies

$$\left(\frac{\partial}{\partial \bar{t}^{\bar{i}}} - \frac{1}{2} C_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \frac{\partial^2}{\partial x^j \partial x^k} - \lambda^2 G_{\bar{i}j} x^j \frac{\partial}{\partial \lambda}\right) e^W = 0$$

$$\left(\frac{\partial}{\partial t^i} + \Gamma^k_{ij} x^j \frac{\partial}{\partial x^k} + \partial_i K \left(\lambda \frac{\partial}{\partial \lambda} - x\frac{\partial}{\partial x} + 1 - \frac{\chi}{24}\right) + \partial_i F_1 \right.$$
$$\left. - \frac{\partial}{\lambda \partial x^i} + \frac{1}{2} C_{ijk} x^j x^k\right) e^W = 0$$

e^W is a section of the line bundle $\mathcal{L}^{\frac{\chi}{24} - 1}$

By noting that the general solution of F_1's anomaly equation is

$$F_1 = \frac{1}{2}\log\det G + \left(\frac{h+1}{2} + 1 - \frac{\chi}{24}\right) K + f_1 + \bar{f}_1$$

where $h = h_{2,1}$ (IIB) and f_1 is holomorphic, Verlinde 0412139
$h_{1,1}$ (IIA)

and multiplying W by $\exp f_1$, one may rewrite the system as

$$\left[\begin{aligned} \partial_{\bar{i}} \Psi_{top} &= \left[\frac{e^{2K}}{2} \bar{C}_{\bar{i}\bar{j}\bar{k}}\, G^{j\bar{j}} G^{k\bar{k}} \frac{\partial^2}{\partial x^i \partial x^j} + G_{ij}\, x^j \frac{\partial}{\partial \lambda^{-1}} \right] \Psi_{top} \\ \left(\nabla_i + \Gamma_{ij}^{k} x^j \frac{\partial}{\partial x^k}\right) \Psi_{top} &= \left[\frac{1}{\lambda}\frac{\partial}{\partial x^i} - \frac{1}{2}\partial_i \log\det G - \frac{1}{2} x^j x^k C_{ijk} \right] \Psi_{top} \end{aligned}\right.$$

where $\Psi_{top} = \exp(f_1)\, W$ is now a section of $\mathcal{L}^{\frac{h+1}{2}}$.

This suggests that $\Psi_{top}(\lambda, x\,;\, t, \bar{t})$ should be viewed as a wave function, obtained by quantizing a finite dimensional phase space $(\lambda, x\,;\, \bar{\lambda}, \bar{x})$ in a "complex" polarization specified by $(t, \bar{t})$. The anomaly equations indicate the canonical transformation experienced by the wave function as the polarization is varied. In particular, the norm

$$\langle \Psi_{top} | \Psi_{top} \rangle = \int \frac{d\lambda\, d\bar{\lambda}\, d^h x\, d^h \bar{x}}{|\lambda|^4}\, e^{\frac{h+1}{2}K} |G|^{1/4}.$$
$$\exp\left[-e^{-K} G_{i\bar{j}}\, x^i \bar{x}^{\bar{j}} + \frac{e^{-K}}{\lambda\bar{\lambda}} \right] \Psi_{top\, t\bar{t}}(\lambda, x)\ \Psi^*_{top\, t\bar{t}}(\bar{\lambda}, \bar{x})$$

is independent of $(t, \bar{t})$.
The two anomaly equations are conjugate with respect to this norm.

This is most easily understood in the B-model: the space $H^3(M,\mathbb{R})$ has a symplectic structure

$$Q = \int_{CY} \delta\gamma \wedge \delta'\gamma$$

If one parametrizes $H^3(M,\mathbb{R})$ using real periods

$$\int_{A^I} \gamma = p^I \qquad \text{then} \quad Q = dp^I \wedge dq_I$$
$$\int_{B_J} \gamma = q_J$$

However, one could instead use the Hodge decomposition

$$\gamma = \frac{1}{2}\left(\frac{\Omega}{\lambda} + x^i \nabla_i \Omega + \bar{x}^{\bar{i}} \bar{\nabla}_{\bar{i}} \bar{\Omega} + \frac{\bar{\Omega}}{\bar{\lambda}} \right)$$
$$\in \quad H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$$

The relation between the two basis is

$$p^I = \mathrm{Re}\left(\frac{X^I}{\lambda} + x^i \nabla_i X^I \right) \qquad \text{since} \quad X^I = \int_{A^I} \Omega$$
$$q_I = \mathrm{Re}\left(\frac{F_I}{\lambda} + x^i \nabla_i F_I \right) \qquad F_I = \int_{B_I} \Omega$$

In this basis, the symplectic form becomes

$$Q = i e^{-K} G_{i\bar{j}}\, dx^i\, d\bar{x}^{\bar{j}} + i e^{-K} d\bar{\lambda}^{-1} \wedge d\lambda$$

Under a variation of the complex structure $\Omega(t,\bar{t})$, the parameters $1/\lambda$, x^i can be adjusted by

$$\bar{\partial}_{\bar{i}}\ 1/\lambda = - G_{\bar{i}j}\, x^j$$
$$\bar{\partial}_{\bar{i}}\ x^k = e^K \bar{C}_{\bar{i}\bar{j}\bar{k}}\, G^{k\bar{k}}$$

so that γ itself remains constant.

Quantum mechanically, the coherent state which diagonalizes x, λ^{-1} experiences a Bogoliubov transformation

$$\bar{\partial}_{\bar{i}} |x,\lambda\rangle = \left(\frac{e^{2K}}{2} C_{\bar{i}\bar{j}\bar{k}}\, G^{k\bar{k}} G^{j\bar{j}} \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^k} - G_{\bar{i}j}\, x^j \frac{\partial}{\partial \lambda^{-1}} \right) |x,\lambda\rangle$$

This is precisely the holomorphic anomaly equation, provided one identifies $\Psi_{top}(x,\lambda;t\bar{t}) = \langle\psi_{top} | x,\lambda\rangle$

where $|\psi_{top}\rangle$ is a state in an abstract Hilbert space, independent of $t,\bar{t}$.

In particular, we may choose to represent it in the real polarization, where $q_I = \frac{\partial}{\partial p^I}$:

Define the "big phase space" variables $x^I = \frac{X^I}{\lambda} + x^i \nabla_i X^I$

so $x^I \sim (\lambda, x^i)$

The relation with real basis is now
$$\left| \begin{array}{l} Re(x^I) = p^I \\ Re(\tau_{IJ}\, x^J) = q_I \end{array}\right.$$

which is solved by $x^I = -i\,(Im\,\tau^{-1})^{IJ}(q_J - \bar{\tau}_{JK}\, p^K)$

Quantum mechanically, $\psi_{top}(x^I)$ is related to $\psi_{top}(p)$ by a Bargmann Ker

$$\Psi_{top}(x^I; t\bar{t}) = \underline{\int dp}\ \Psi_{top}(p)\ exp\left[\frac{i}{2}\left(p^I \bar{\tau}_{IJ}{}^J + p^I(\tau-\bar{\tau})x^J\right.\right.$$
$$\left. - \frac{1}{4}(\tau-\bar{\tau})\, x^I x^J \right]$$

There is an interesting dynamical system leading to a phase space $H_3(M,\mathbb{R})$ the NS5-brane in IIA on CY - or equivalently the M5-brane in M theory. Since it carries a self dual 3-form flux, its phase space is $H_3(M,\mathbb{R})$. Its partition function may be obtained by factorizing that of a general 3-form flux:

In this set up, p^I is viewed as the flux on the NS5 brane, and x^I are the background values of the 3 form. The full partition function appears to involve a theta series over p^I for the classical part, as the infinite series of topological amplitudes $exp\left[\sum_{g\geq 1} \mathcal{F}_g(t,\bar{t})\,\lambda^{2g-2}\right]$ for the quantum part

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As we shall discuss later, there is another natural set-up where the topological string amplitude seems to play the rôle of a wave function: radial quantization of $^{\text{BPS}}$ black holes, or equivalently, the quantum attractor flow.

Before closing this chapter, one final observation: since $\Psi_{top}(x^I; t, \bar{t})$ is interpreted as a wave function, it follows that under electric magnetic duality $x^I \to F_I$, $F_I \to -x^I$, it transforms by Fourier under all variables x^I.

This is reminiscent of Kontsevitch's "Very Wild Ones" Arbeitstagung,9

$$\text{Fourier}\left(\exp F_0\right) \overset{?}{=} \exp\left(\text{Legendre } F_0\right)$$

This "Master equation" is at least verified by all the very special supergravities, where

Kazhdan, Etingov, Polischchuk

$$F_0 = \frac{N(X)}{X^0}$$

$N(X)$ = norm of a degree 3 Jordan algebra