Five-brane instantons, topological wave-functions and hypermultiplets

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Motivation

- The study of the vector multiplet moduli space (VM) and BPS spectrum in string vacua with $N = 2$ supersymmetries in $D = 4$ has had tremendous applications in mathematics and physics: classical mirror symmetry, Gromov-Witten invariants, black hole precision counting, etc...

$$\mathrm{IIB}/\hat{\mathcal{X}} \;\Big|\; \mathrm{IIA}/\mathcal{X} \;\Big|\; \mathrm{Het}/K_3 \times T^2$$

- Understanding the hypermultiplet moduli space (HM) may be even more rewarding: a quantum extension mirror symmetry beyond classical and homological mirror symmetry, new checks of Het/II duality, new geometric invariants, richer automorphic properties...

Hypers = Vectors

- Upon circle compactification to $D=3$, VM and HM become two sides of the same coin, exchanged by T-duality along the circle:

$$\mathrm{IIB}/\hat{\mathcal{X}} \times S^1 \;\Big|\; \mathrm{IIA}/\mathcal{X} \times S^1 \;\Big|\; \mathrm{Het}/K_3 \times T^3$$

- In $D=3$, both VM and HM are quaternion-Kähler manifolds:

$$\begin{aligned} \mathrm{HM}_3 &= \mathrm{HM}_4 \;, \\ \mathrm{VM}_3 &= \mathrm{c-map}(\mathrm{VM}_4) + 1-\mathrm{loop} + \mathcal{O}(e^{-r}) + \mathcal{O}(e^{-r^2}) \end{aligned}$$

  VM_3 includes, in addition to VM_4, the radius of the circle, the electric and magnetic holonomies of the $D=4$ Maxwell fields, and the NUT potential, dual to the Kaluza-Klein gauge field in $D=3$.

Instantons = Black holes + KKM

- The $\mathcal{O}(e^{-r})$ corrections come from BPS states in $D=4$, whose Euclidean wordline winds around the circle: thus VM_3 encodes the $D=4$ BPS black hole spectrum, with chemical potentials for every electric and magnetic charges, consistently with chamber dependence !

Gunaydin Neitzke BP Waldron, Gaiotto Moore Neitzke, Kontsevich Soibelman

- The $\mathcal{O}(e^{-r^2})$ corrections come from gravitational instantons of the form $\mathrm{TN}_k \times \mathcal{Y}$ ($\mathcal{Y} = \hat{\mathcal{X}}, \mathcal{X}, K_3 \times T^2$), i.e. Kaluza-Klein monopoles (in Lorentzian signature, these would have closed timelike curves).

Changing currencies

- On the flip side of the coin, in type II currency, the corrections to HM_4 instead originate from Euclidean D-branes ($\mathcal{O}(e^{-1/g_s})$) and Euclidean NS5-branes ($\mathcal{O}(e^{-1/g_s^2})$).

Becker Becker Strominger

- The reference currency for hypermultiplets is Het/$K_3 \times T^2$: since the heterotic string coupling is a VM, *HM_4 is exact at heterotic string tree-level !* (though it receives 'one-loop' and nonperturbative α' corrections)

Aspinwall

- Recent progress has instead occurred on the type II side, combining S-duality and mirror symmetry with improved twistor techniques. D-brane instantons are essentially under control (see Alexandrov's talk) *except for convergence issues.*
- Five-brane instantons will be the main subject of this talk.

Alexandrov Persson BP, to appear

Automorphy I

- The equivalent constructions

$$\mathrm{IIB}/\hat{\mathcal{X}} \times S^1 \;\Big|\; M/\mathcal{X} \times T^2 \;\Big|\; M/K_3 \times K_3'$$

suggest that for any CY threefold $\mathcal{X}$, HM should have an isometric action of $SL(2,\mathbb{Z})$ “S-duality". This has been used to determine D(-1)-D1 instanton corrections, from known tree level and one-loop corrections.

Robles-Llana Rocek Saueressig Theis Vandoren

- Care must be exercised, since dualities in $N = 2$ vacua tend to be broken to inite index subgroups, e.g. $\Gamma(2)$ in Seiberg-Witten theory with no flavor. As we shall see, fractional charge shifts in the presence of D3-D5-NS5 will force us to relax some of the $SL(2,\mathbb{Z})$ generators.

Automorphy II

- Ignoring these subtleties, combinations of S-duality with the monodromy group $\mathrm{Mon}(X) \subset Sp(2n,\mathbb{Z})$ and the Heisenberg group of shifts of the electric/magnetic/NUT potentials suggest that HM should have a much larger discrete group of isometries, an arithmetic subgroup of $Sp(2n+2,\mathbb{Z})$, that remains to be identified.
- Some qualitative insights into five-brane instanton corrections were gained recently by assuming invariance under $SL(3,\mathbb{Z})$ for non-rigid $\mathcal{X}$, or the Picard subgroup $SU(2,1,\mathbb{Z}[\sqrt{-d}])$ for rigid $\mathcal{X}$ with complex multiplication. I shall not discuss these works in detail here, although they played an important rôle in shaping the approach below

Persson BP; Bao Kleinschmidt Nilsson Persson BP

Topological wave functions I

Finally, let me get to the “topological wave function" part of the title:

- the key point is that in a sector with k five-branes (or KKM) discrete shifts of the RR-axions (or electromagnetic potentials) no longer commute. Instead, they generate a Heisenberg group:

$$T_H \cdot T_{H'} \cdot T_{-H} \cdot T_{-H'} = e^{\mathrm{i}k\langle H,H'\rangle} \ .$$

  Thus, only a Lagrangian subspace of the lattice Γ of electromagnetic charges can be diagonalized simultaneously, leading to wave-function behavior under changes of polarization.

- The same behavior is known to occur for the topological string amplitude $\Psi_{\rm top}$. By the GW/DT relation, $\Psi_{\rm top}$ counts D6-D2-D0 bound states with [D6]=1, or equivalently D5-D1-D(-1) instantons with [D5]=1. By S-duality, $\Psi_{\rm top}$ should also govern the contribution of a single NS5-brane instanton, bound to arbitrary D-instantons.

Topological wave functions II

- The study of the HM gives a sharp formulation of a relation between the five-brane partition function and topological string amplitude which was anticipated long ago.

Dijkgraaf Verlinde Vonk; Kapustin; Marino Minasian Moore Strominger

- Optimistically, one may hope that HM also allows for a sharp formulation of the OSV conjecture...

Outline

1. Introduction
2. Topology of the HM moduli space
3. Qualitative aspects of five-brane corrections
4. Mirror symmetry, S-duality and (p, k) fivebranes
5. Conclusion

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The perturbative metric I

- The HM moduli space in type IIA compactified on the CY 3-fold (family) $\mathcal{X}$ is a quaternion-Kähler manifold $\mathcal{M} = \mathcal{Q}_c(\mathcal{X})$ of real dimension $2b_3(\mathcal{X}) = 4(h_{2,1}+1)$. It encodes the 4D dilaton $r \equiv e^\phi \sim 1/g^2_{(4)}$, complex structure of $\mathcal{X}$, RR-field C and NS axion σ.
- In the weak coupling limit $r \to \infty$, the quaternion-Kähler metric on $\mathcal{M}$ is given, to all orders in $1/r$, by

$$ds^2_{\mathcal{M}} = \frac{r+2c}{r^2(r+c)}\,\mathrm{d}r^2 + \frac{4(r+c)}{r}\,\mathrm{d}s^2_{\mathcal{SK}} + \frac{\mathrm{d}s^2_T}{r} + \frac{2\,c}{r^2}\,e^{\mathcal{K}}\,|X^\Lambda \mathrm{d}\tilde\zeta_\Lambda - F_\Lambda \mathrm{d}\zeta^\Lambda|^2 + \frac{r+c}{16r^2(r+2c)} D\sigma^2\,.$$

where $c = -\frac{\chi_{\mathcal{X}}}{192\pi}$ originates from a one-loop correction. Note the curvature singularity at $r = -2c$ when $\chi_{\mathcal{X}} > 0$!

Cecotti Girardello Ferrara; Ferrara Sabharwal

Antoniadis Minasian Theisen Vanhove; Robles-Llana Saueressig Vandoren

Notations I

Complex structure moduli $\Omega = (X, F)$ and RR-axions $C = (\zeta, \tilde{\zeta})$:

$$X^\Lambda = \int_{\mathcal{A}^\Lambda} \Omega\,, \quad F_\Lambda = \int_{\mathcal{B}_\Lambda} \Omega\,, \quad \zeta^\Lambda = \int_{\mathcal{A}^\Lambda} C, \quad \tilde{\zeta}_\Lambda = \int_{\mathcal{B}_\Lambda} C\,.$$

Special Kähler metric on complex structure moduli space $\mathcal{M}_c(\mathcal{X})$

$$\mathrm{d}s^2_{\mathcal{SK}} = \partial\bar{\partial}\mathcal{K}\,, \qquad \mathcal{K} = -\log[\mathrm{i}(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda)]$$

Kahler metric on intermediate Jacobian T:

$$T = \frac{H^3(\mathcal{X}, \mathbb{R})}{H^3(\mathcal{X}, \mathbb{Z})}\,, \quad \mathrm{d}s^2_T = -\frac{1}{2}(\mathrm{d}\tilde{\zeta}_\Lambda - \bar{\mathcal{N}}_{\Lambda\Lambda'}\mathrm{d}\zeta^{\Lambda'})\mathrm{Im}\mathcal{N}^{\Lambda\Sigma}(\mathrm{d}\tilde{\zeta}_\Lambda - \mathcal{N}_{\Sigma\Sigma'}\mathrm{d}\zeta^{\Sigma'})$$

Horizontal one-form for NS axion:

$$D\sigma = \mathrm{d}\sigma + \tilde{\zeta}_\Lambda \mathrm{d}\zeta^\Lambda - \zeta^\Lambda \mathrm{d}\tilde{\zeta}_\Lambda + 8c\mathcal{A}_K\,, \qquad \mathcal{A}_K = \frac{\mathrm{i}}{2}(\mathcal{K}_a \mathrm{d}z^a - \mathcal{K}_{\bar{a}}\mathrm{d}\bar{z}^{\bar{a}})$$

Topology of the RR moduli space I

- At least at weak coupling, HM is foliated by hypersurfaces $\mathcal{C}(r)$ of constant string coupling.
- Quotienting along the NS axion σ, $\mathcal{C}(r)/\partial_\sigma$ reduces to a torus bundle over $\mathcal{M}_c(\mathcal{X})$, with fiber T parametrizing the RR field C. Large gauge transformations require that $(\zeta^\Lambda, \tilde{\zeta}_\Lambda)$ have integer periodicities.
- This periodicity is consistent with the fact that Euclidean D2-branes wrapping a SLAG submanifold $\mathcal{L}$ induce corrections of the form

$$\delta \mathrm{d}s^2|_{D2} \sim \exp\left(-8\pi e^{\phi/2}|Z_\gamma| - 2\pi \mathrm{i}(q_\Lambda \zeta^\Lambda - p^\Lambda \tilde{\zeta}_\Lambda)\right) ,$$

where $Z_\gamma \equiv e^{\mathcal{K}/2}(q_\Lambda X^\Lambda - p^\Lambda F_\Lambda)$ is the central charge, and p^Λ, q_Λ label the integer homology class $[\mathcal{L}] = q_\Lambda \mathcal{A}^\Lambda - p^\Lambda \mathcal{B}_\Lambda \in H_3(\mathcal{X}, \mathbb{Z})$.

Topology of the NS axion I

- We refer to the torus bundle $\mathcal{C}(r)/\partial_\sigma$ as the "intermediate Jacobian" $\mathcal{J}(\mathcal{X})$ of (the CY family) $\mathcal{X}$. Under a monodromy M in $\mathcal{M}_c(\mathcal{X})$, T changes by symplectic rotations $C \mapsto \rho(M)C$

- Assuming (as we shall justify later from S-duality) that σ is periodic with period 2, the horizontal one-form $D\sigma$ implies that $e^{-\mathrm{i}\pi\sigma}$ parametrizes the fiber of a circle bundle over $\mathcal{J}(\mathcal{X})$, with first Chern class

$$c_1[\mathcal{C}(r)] = -\omega_T - \frac{\chi_{\mathcal{X}}}{24}\,\omega_c\,,$$

where ω_T, ω_c are the Kähler forms on T and $\mathcal{M}_c(\mathcal{X})$, resp.

Topology of the NS axion II

- Both these terms follow by dimensional reduction from the topological coupling in $D=10$ type IIA supergravity:

$$\int_{\mathcal{Y}} \left(\frac{1}{6} B \wedge \mathrm{d}B \wedge \mathrm{d}B - B \wedge I_8 \right) , \quad I_8 = \frac{1}{48}(p_2 - \frac{1}{4} p_1^2)$$

Indeed, on a complex manifold $\mathcal{Y}$,

$$B \wedge I_8 = \frac{1}{24} B \wedge \left[c_4 - c_1 \left(c_3 + \frac{1}{8} c_1^3 - \frac{1}{2} c_1 c_2 \right) \right] .$$

Integrating the term in parenthesis on the CY threefold $\mathcal{X}$ produces a coupling $\frac{\chi_{\mathcal{X}}}{24} B \wedge \omega_c$ in $\mathbb{R}^4$. Dualizing the two-form B into σ produces the $\chi_{\mathcal{X}}$ dependent correction to $D\sigma$.

Topology of the NS axion III

- The tree-level term in $D\sigma = \mathrm{d}\sigma + \tilde{\zeta}_\Lambda \mathrm{d}\zeta^\Lambda - \zeta^\Lambda \mathrm{d}\tilde{\zeta}_\Lambda + \dots$ implies that translations on T must be accompanied with shifts of σ,

$$T_{(H,p)}: \quad \begin{cases} \zeta^\Lambda & \mapsto \quad \zeta^\Lambda + \eta^\Lambda \\ \tilde{\zeta}_\Lambda & \mapsto \quad \tilde{\zeta}_\Lambda + \tilde{\eta}_\Lambda \\ \sigma & \mapsto \quad \sigma + 2p - \tilde{\eta}_\Lambda \zeta^\Lambda + \eta^\Lambda \tilde{\zeta}_\Lambda + c(\eta, \tilde{\eta}) \end{cases}$$

where $H \equiv (\eta^\Lambda, \tilde{\eta}_\Lambda) \in \mathbb{Z}^{b_3}$, $p \in \mathbb{Z}$ and $c(H)$ defines a quadratic refinement of the intersection form on $H^3(\mathcal{X}, \mathbb{Z})$,

$$\sigma(H + H') = (-1)^{\langle H, H' \rangle} \sigma(H)\, \sigma(H')\,, \qquad \sigma(H) \equiv (-1)^{c(H)}$$

such that $T_{(H,p)}$ satisfies the Heisenberg group rule

$$T_{(H,p)} T_{(H',p')} = T_{(H+H', p+p'+\frac{1}{2}\langle H, H' \rangle)}\ .$$

Topology of the NS axion IV

- Quadratic refinements can be parametrized by characteristics $\Theta = (\theta, \phi) \in \{0, \frac{1}{2}\}^{b_3}$,

$$c(\eta, \tilde{\eta}) = -\eta^\Lambda \tilde{\eta}_\Lambda + 2\tilde{\eta}_\Lambda \theta^\Lambda - 2\eta^\Lambda \phi_\Lambda \ .$$

- Moreover, the one-loop correction in $D\sigma$ implies that under rescalings of the holomorphic 3-form $\Omega \mapsto e^f \Omega$,

$$\sigma \mapsto \sigma + \frac{\chi_{\mathcal{X}}}{24\pi} \mathrm{Im}(f)$$

- These statements can be summarized by saying that $e^{-i\pi\sigma}$ transforms as a section of $\mathcal{L}^{-\chi_{\mathcal{X}}/24} \otimes \mathcal{L}_\Theta$, where $\mathcal{L}$ is the line bundle over $\mathcal{M}_c$ where Ω is valued, and $\mathcal{L}_\Theta$ is the Theta line bundle over T with characteristics Θ.

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Qualitative aspects of five-brane corrections I

- Instanton corrections from k fivebranes (or, in VM language, k KKM) must depend on the NS-axion by an overall factor $e^{-\mathrm{i}\pi k\sigma}$,

$$\delta \mathrm{d}s^2|_{\mathrm{NS5}} \sim Z^{(k)}(\phi, z^a, C)\, e^{-\mathrm{i}\pi k\sigma}$$

- In order to be well defined under large C-gauge transformations and rescalings of Ω, $Z^{(k)}$ must be a section of the line bundle $\mathcal{L}^{\chi_{\mathcal{X}}/24} \otimes \mathcal{L}_{-\Theta}$.
- Remarkably, the partition function of a chiral five-brane on $\mathcal{X}$ is known to be a holomorphic section of $\mathcal{L}_{-\Theta}$ for fixed $\mathcal{X}$. We shall argue that it also transforms as $\mathcal{L}^{\chi_{\mathcal{X}}/24}$ under variations of $\mathcal{X}$.

Witten; Dijkgraaf Verlinde Vonk;Henningson Nilsson Salomonson; Belov Moore; ...

Gaussian fivebrane partition function I

- In the weak coupling limit, the partition function of a chiral five-brane can be obtained by holomorphic factorization of a non-chiral 3-form $H = \mathrm{d}\mathcal{B}$ on $\mathcal{X}$, with Gaussian action such that the bulk 3-form C couples only to the self-dual part of H,

$$S(H, C) = \pi \int_{\mathcal{X}} (H - C) \wedge \star (H - C) - \mathrm{i}\pi \int_{\mathcal{X}} C \wedge H \,.$$

- The sum splits over topological sectors $[H] = m_\Lambda \mathcal{A}^\Lambda - n^\Lambda \mathcal{B}_\Lambda$. Neglecting quantum fluctuations around $S([H], C)$, inserting a power of the quadratic refinement σ_Θ, and Poisson resumming over m_Λ, the partition function decomposes into

$$\sum_{H \in H^3(\mathcal{X},\mathbb{Z})} [\sigma_\Theta(H)]^k e^{-kS(H,C)} = N \sum_{\mu \in \Gamma_m / k\Gamma_m} Z^{(k)}_{\Theta,\mu+\mu'}(\mathcal{N}, 0)\, \overline{Z^{(k)}_{\Theta,\mu-\mu'}(\mathcal{N}, C)}$$

Gaussian fivebrane partition function II

- Here $\mathcal{Z}$ is the Siegel theta series of rank $b_3(\mathcal{X})$, level $k/2$

$$\mathcal{Z}^{(k)}_{\Theta,\mu}(\mathcal{N}, C) = \sum_{n \in \Gamma_m + \mu + \theta} \mathrm{e}^{\frac{k}{2}(\zeta^\Lambda - n^\Lambda)\bar{\mathcal{N}}_{\Lambda\Sigma}(\zeta^\Sigma - n^\Sigma) + k(\tilde{\zeta}_\Lambda - \phi_\Lambda)n^\Lambda + \frac{k}{2}(\theta^\Lambda \phi_\Lambda - \zeta^\Lambda \tilde{\zeta}_\Lambda)},$$

  where $\mathrm{e}^x \equiv e^{2\pi \mathrm{i} x}$, and N is a C-independent factor.
- Under large gauge transformations, the variation of $\mathcal{Z}$

$$\mathcal{Z}^{(k)}_{\Theta,\mu}(\mathcal{N}, C + H) = (\sigma_\Theta(H))^k \, \mathrm{e}^{\frac{k}{2}(n^\Lambda \tilde{\zeta}_\Lambda - m_\Lambda \zeta^\Lambda)} \mathcal{Z}^{(k)}_{\Theta,\mu}(\mathcal{N}, C)$$

  precisely cancels the variation of $e^{-\mathrm{i}\pi\sigma}$!
- The Gaussian approximation breaks down when $|H| \sim 1/g_s$, and $S(H, C)$ should be replaced by the non-linear five-brane action. Presumably this does not affect the above periodicity property.

Bandos et al; Aganagic et al; Cederwall Nilsson Sundell

Gaussian fivebrane partition function III

- The dependence on the metric of $\mathcal{X}$ is notoriously subtle, as a consequence of the $B \wedge I_8$ topological term.

Witten; Belov Moore

- On the other hand, the topological string amplitude, which should be related to $Z^{(1)}$ by S-duality, is known to be a section of $\mathcal{L}^{\chi_{\mathcal{X}}/24-1}$.

Bershadsky Cecotti Ooguri Vafa

- This strongly suggests that the corrections from a single five-brane should be given by a theta series built on the topological string amplitude:

$$\mathcal{Z}^{(k)}_{\Theta,\mu}(\mathcal{N}, C) = \sum_{n \in \Gamma_m + \mu + \theta} \Psi_{\rm top}(\zeta^\Lambda - n^\Lambda)\, \mathrm{e}^{k(\tilde\zeta_\Lambda - \phi_\Lambda) n^\Lambda + \frac{k}{2}(\theta^\Lambda \phi_\Lambda - \zeta^\Lambda \tilde\zeta_\Lambda)} ,$$

where $\Psi_{\rm top}(\zeta^\Lambda)$ is the top amplitude in the real polarization.

Gaussian fivebrane partition function IV

- For $k > 1$, we expect that five-brane corrections will be governed by a non-abelian generalization of $\Psi_{\rm top}$ which counts rank k Donaldson-Thomas invariants.
- Our strategy will be to pass to type IIB string theory on the mirror $\hat{\mathcal{X}}$, and use S-duality to relate D5-brane instantons to (p, k) fivebranes.

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Mirror symmetry I

- In type IIB/$\hat{\mathcal{X}}$, the perturbative metric on HM takes the same form as before, where $z^a = b^a + \mathrm{i} t^a = X^a/X^0$ are now the Kähler moduli of $\hat{\mathcal{X}}$, and $(\zeta^0, \zeta^\Lambda, \tilde{\zeta}_\Lambda, \tilde{\zeta}_0)$ label the periods of the RR field $A = A^{(0)} + A^{(2)} + A^{(4)} + A^{(6)} \in H^{\mathrm{even}}(\hat{\mathcal{X}}, \mathbb{R})$.
- Near the infinite volume point, the prepotential governing $\mathcal{M}_K(\hat{\mathcal{X}})$ is

$$F(X) = -\frac{N(X^a)}{X^0} + \frac{1}{2} A_{\Lambda\Sigma} X^\Lambda X^\Sigma + \chi_{\hat{\mathcal{X}}} \frac{\zeta(3)(X^0)^2}{2(2\pi\mathrm{i})^3} + F_{\mathrm{GW}}(X)$$

where $N(X^a) \equiv \frac{1}{6} \kappa_{abc} X^a X^b X^c$, and $A_{\Lambda\Sigma}$ is a constant, real symmetric matrix, defined up to integer shifts.

Mirror symmetry II

- D-instantons are now Euclidean D5-D3-D1-D(-1), described mathematically by rank p^0 coherent sheaves F on $\mathcal{X}$. Their charges (p^Λ, q_Λ) can be expressed in terms of the Chern classes by matching the central charge,

$$q_\Lambda X^\Lambda - p^\Lambda F_\Lambda = e^{-\mathcal{K}/2} Z_\gamma = \int_{\hat{\mathcal{X}}} e^{-(B+\mathrm{i}J)}\, \mathrm{ch}(F)\, \sqrt{\mathrm{Td}(\hat{\mathcal{X}})}$$

leading to

$$p^0 = \mathrm{rk}(F)\ , \qquad p^a = \int_{\gamma^a} c_1(F)$$

$$q_a = \int_{\gamma_a} \left[c_2(F) - \frac{1}{2} c_1^2(F) \right] + p^0 \left(A_{0a} - \frac{c_{2,a}}{24} \right) + A_{ab} p^b\ ,$$
$$q_0 = \int_{\hat{\mathcal{X}}} \mathrm{ch}(F)\, \mathrm{Td}(\hat{\mathcal{X}}) + p^a \left(A_{0a} - \frac{c_{2,a}}{24} \right) + A_{00} p^0 .$$

Douglas Reinbacher Yau, revisited

Mirror symmetry III

- Noting that $\int_{\hat{\mathcal{X}}} \mathrm{ch}(F)\,\mathrm{Td}(\hat{\mathcal{X}})$ is integer, being the index of the Dirac operator coupled to F, we see that the charges q_Λ are integer iff

$$A_{00} \in \mathbb{Z}\,, \quad A_{0a} \in \frac{c_{2,a}}{24} + \mathbb{Z}\,,$$
$$\frac{1}{2}\,\kappa_{abc} p^b p^c - A_{ab} p^b \in \mathbb{Z} \quad \text{for } \forall p^a \in \mathbb{Z}\,.$$

  E.g. for the quintic, $\kappa_{aaa} = 5, A_{0a} = 25/12, A_{aa} = -11/2, A_{00} = 0$.

- The matrix $A_{\Lambda\Sigma}$ may be set to zero by a non-integer symplectic transformation, leading to non integer electric charges q'_Λ,

$$q'_\Lambda = q_\Lambda - A_{\Lambda\Sigma} p^\Sigma\,, \quad \tilde{\zeta}'_\Lambda = \tilde{\zeta}_\Lambda - A_{\Lambda\Sigma}\zeta^\Lambda\,, \quad F' = F - \frac{1}{2} A_{\Lambda\Sigma} X^\Lambda X^\Sigma$$
$$q'_a \in \mathbb{Z} - \frac{p^0}{24}\,c_{2,a} - \frac{1}{2}\kappa_{abc} p^b p^c\,, \qquad q'_0 \in \mathbb{Z} - \frac{1}{24}\,p^a c_{2,a}\,,$$

  Note that $q_\Lambda \zeta^\Lambda - p^\Lambda \tilde{\zeta}_\Lambda = q'_\Lambda \zeta^\Lambda - p^\Lambda \tilde{\zeta}'_\Lambda$.

S-duality in twistor space I

- At zero coupling / infinite volume, the HM metric admits an isometric action of $SL(2,\mathbb{R})$, corresponding to type IIB S-duality in 10 dimensions.
- Any isometry of a QK manifold $\mathcal{M}$ can be lifted to a holomorphic action on its twistor space $\mathcal{Z}$ (a $\mathbb{P}^1$ bundle over $\mathcal{Z}$ with a canonical complex contact structure, see Alexandrov's talk). Let t be a complex coordinate on $\mathbb{P}^1$ and define the following Darboux coordinates on $\mathcal{Z}$:

$$\begin{aligned} \xi^\Lambda &= \zeta^\Lambda + \frac{\tau_2}{2}\left(t^{-1} z^\Lambda - t\,\bar{z}^\Lambda\right), \\ \rho'_\Lambda &= \tilde{\zeta}'_\Lambda + \frac{\tau_2}{2}\left(t^{-1} F'_\Lambda(z) - t\,\bar{F}'_\Lambda(\bar{z})\right), \\ \tilde{\alpha} &= \sigma + \frac{\tau_2}{2}\left(t^{-1} W(z) - t\,\bar{W}(\bar{z})\right) + \frac{\mathrm{i}\chi_{\mathcal{X}}}{24\pi}\log t\,, \end{aligned}$$

where $\tau_2^2 = 16 e^{(\phi+\mathcal{K})} - \frac{\chi_{\hat{\mathcal{X}}}}{48\pi}$, $W(z) \equiv F'_\Lambda(z)\zeta^\Lambda - z^\Lambda \tilde{\zeta}'_\Lambda$, such that

$$Dt = \mathrm{d}t + p_+ - \mathrm{i}p_3 t + p_- t^2 \propto \mathrm{d}\tilde{\alpha} + \xi^\Lambda \mathrm{d}\rho'_\Lambda - \rho'_\Lambda \mathrm{d}\xi^\Lambda$$

S-duality in twistor space II

- $SL(2,\mathbb{R})$ acts on the above Darboux coordinates via

$$
\begin{aligned}
&\xi^0 \mapsto \frac{a\xi^0+b}{c\xi^0+d}\,, \qquad \xi^a \mapsto \frac{\xi^a}{c\xi^0+d}\,,\\
&\tilde\xi'_a \mapsto \tilde\xi'_a + \frac{\mathrm{i}\,c}{4(c\xi^0+d)}\kappa_{abc}\xi^b\xi^c\,,\\
&\begin{pmatrix}\tilde\xi'_0\\ \alpha'\end{pmatrix} \mapsto \begin{pmatrix} d & -c\\ -b & a\end{pmatrix}\begin{pmatrix}\tilde\xi'_0\\ \alpha'\end{pmatrix}\\
&\qquad + \frac{\mathrm{i}}{12}\kappa_{abc}\xi^a\xi^b\xi^c \begin{pmatrix} c^2/(c\xi^0+d)\\ -[c^2(a\xi^0+b)+2c]/(c\xi^0+d)^2\end{pmatrix}\,,
\end{aligned}
$$

where $\tilde\xi'_\Lambda = \frac{\mathrm{i}}{2}\rho'_\Lambda, \alpha' = (\tilde\alpha + \xi^\Lambda\rho'_\Lambda)/(4\mathrm{i})$.

- Rk: the transformations of $\tilde\xi'_\Lambda, \alpha'$ can be rewritten as $\Xi_a \mapsto \frac{\Xi_a}{(c\xi^0+d)^2}, \Xi_0 \mapsto \frac{\Xi_0}{(c\xi^0+d)^3}$ if so desired.

S-duality in twistor space III

- $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ exchanges $\tilde{\zeta}'_0$ with $-\frac{1}{2}\sigma + \dots$, warranting our earlier claim about mod 2 periodicity of the NS-axion σ.
- $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty$ acts by

$$\xi^0 \mapsto \xi^0 + b\,, \quad (\xi^a, \tilde{\xi}'_a) \mapsto (\xi^a, \tilde{\xi}'_a)\,, \quad \alpha' \mapsto \alpha' - b\tilde{\xi}'_0$$

- On the other hand, the Heisenberg group of isometries acts holomorphically on $\mathcal{Z}$ by

$$T_{(H,p)} \;: \left(\xi^\Lambda, \rho_\Lambda, \tilde{\alpha}\right) \mapsto \Big(\xi^\Lambda + \eta^\Lambda, \rho_\Lambda + \tilde{\eta}_\Lambda, \\ \tilde{\alpha} + 2p - \left[\tilde{\eta}_\Lambda \xi^\Lambda - \eta^\Lambda \rho_\Lambda\right] - \{\eta^\Lambda \tilde{\eta}_\Lambda - 2\tilde{\eta}_\Lambda \theta^\Lambda + 2\eta^\Lambda \phi_\Lambda\}\Big)\,,$$

where $\eta^\Lambda, \tilde{\eta}_\Lambda, p \in \mathbb{Z}$.

S-duality in twistor space IV

- Unless A_{0a} is integer, the S-duality action $\xi^0 \mapsto \xi^0 + b$ is not the same as the Heisenberg shift $\xi^0 \mapsto \xi^0 + \eta^0$ for $b = \eta^0$! the two differ by a shift $\tilde{\xi}_a \mapsto \tilde{\xi}_a + b\, c_{0a}/24$, which is a fraction of the periodicity expected from charge quantization.
- Assuming that the full $SL(2,\mathbb{Z})$ was preserved at the quantum level, Robles-Llana et al constructed an Eisenstein series which unifies the tree-level $\zeta(3)\chi$ and one-loop $\zeta(2)\chi$ corrections and predicts an infinite series of D(-1)-instanton corrections.
- Under the same assumption, we shall construct a Poincaré series from the known form of D5-D3-D1-D(-1) corrections to $\mathcal{Z}$ and obtain the contributions from k five branes.
- For brevity, we mostly ignore quadratic refinements, characteristics, fractional charge shifts, etc.

A Poincaré series for five-brane instantons I

- For fixed D5-brane charge $p^0 \neq 0$, the D-instanton corrections to $\mathcal{Z}$ are formally encoded in an holomorphic section of $H^1(\mathcal{Z}, \mathcal{O}(2))$ (see Alexandrov's talk) – here and below $\gamma = (p^0, p^a, q_a, q_0)$:

$$H(p^0) = -\frac{\mathrm{i}}{8\pi^2} \sum_{p^a, q_a, q_0} \sigma(\gamma)\, \tilde{\Omega}_\gamma\, e^{2\pi \mathrm{i}(p^\Lambda \rho'_\Lambda - q'_\Lambda \xi^\Lambda)} \,,$$

where $\tilde{\Omega}_\gamma$ are rational numbers, related to the integer-valued generalized Donaldson-Thomas invariants by

$$\tilde{\Omega}(\gamma) = \sum_{d|\gamma} \frac{1}{d^2}\, \Omega(\gamma/d)\,, \qquad \Omega(\gamma) = \sum_{d|\gamma} \frac{1}{d^2}\, \mu(d)\, \tilde{\Omega}(\gamma/d),$$

A Poincaré series for five-brane instantons II

- $H(p^0)$ is invariant under $\Gamma_\infty \subset SL(2,\mathbb{Z})$. It can be made invariant under the full $SL(2,\mathbb{Z})$ by summing over $\Gamma_\infty \backslash SL(2,\mathbb{Z})$:

$$H_{\rm tot} = \sum_{p^0} \sum_{(c,d)=1} \delta \cdot H(p^0)\ , \qquad \delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\ .$$

- Each term can be interpreted as a (p,k) five-brane instanton, where $(p,k) = p^0(d,-c)$. Letting $n^0 = p/k, n^a = p^a/k$, and using the invariance of $\tilde{n}_\gamma$ under spectral flow

$$p^0 \mapsto p^0\ , \quad p^a \mapsto p^a + \epsilon^a p^0\ , \quad q'_a \mapsto q'_a - \kappa_{abc} p^b \epsilon^c - \frac{1}{2} p^0 \kappa_{abc} \epsilon^b \epsilon^c\ ,$$

$$q'_0 \mapsto q'_0 - \epsilon^a q'_a + \frac{1}{2} \kappa_{abc} p^a \epsilon^b \epsilon^c + \frac{1}{6} p^0 \kappa_{abc} \epsilon^a \epsilon^b \epsilon^c\ ,$$

A Poincaré series for five-brane instantons III

the sum can be rewritten as a non-Gaussian theta series

$$H^{(k)}_{\rm NS5}(\xi,\tilde\xi,\alpha) = \sum_{\substack{\mu\in(\Gamma_m/|k|)/\Gamma_m \\ n\in\Gamma_m+\mu+\theta}} H_{k,\mu}\left(\xi^\Lambda - n^\Lambda\right)\, \mathrm{e}^{kn^\Lambda(\rho_\Lambda-\phi_\Lambda)-\frac{k}{2}\,(\tilde\alpha+\xi^\Lambda\rho_\Lambda)},$$

where, up to subtle phases,

$$H_{k,\mu}(\xi^\Lambda) = \sum_{q_\Lambda\in\Gamma_e} \tilde\Omega(\gamma)\, \mathrm{e}^{-\frac{k\,N(\xi^a)}{\xi^0}+\frac{k}{2}A_{\Lambda\Sigma}\xi^\Lambda\xi^\Sigma+\frac{p^0}{k}\frac{\hat q_a\xi^a}{\xi^0}+(\frac{p^0}{k})^2\frac{\hat q_0}{\xi^0}}.$$

In this expression, $\gamma=(p^0,k\mu^a,q_a,q_0)$ and $\hat q_\Lambda$ are the spectral flow invariants

$$\hat q_a = q'_a+\frac{k^2}{2p^0}\kappa_{abc}\mu^b\mu^c\,,\qquad \hat q_0 = q'_0+\frac{k}{p^0}\mu^a q'_a+\frac{k^3}{3(p^0)^2}\kappa_{abc}\mu^a\mu^b\mu^c\,,$$

Single fivebrane and topological string amplitude I

- For $k = 1$, μ can be set to 0. Using the GW/DT relation

$$e^{F_{\rm hol}(z^a,\lambda)} = e^{-\frac{(2\pi i)^3}{\lambda^2}(N(z^a) - \frac{1}{2}A_{\Lambda\Sigma}z^\Lambda z^\Sigma) - \frac{2\pi i}{24}c_{2,a}z^a} [M(e^{-\lambda})]^{-\frac{\chi_{\hat{\mathcal{X}}}}{2}}$$
$$\sum_{Q_a,J} (-1)^{2J} N_{DT}(Q_a, 2J)\, e^{-2\lambda J + 2\pi i Q_a z^a} ,$$

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where $M(q) = \prod(1-q^n)^{-n}$ is the Mac-Mahon function, and $N_{DT}(Q_a, 2J)$ are the (ordinary) DT invariants, and the relation

$$e^{F_{\rm hol}(z^a,\lambda)} \sim e^{-f_1(\xi^a/\xi^0)} \left(\xi^0\right)^{\frac{\chi}{24}-1} \Psi^{\rm top}_{\mathbb{R}}(\xi^\Lambda) , \quad \lambda = \frac{2\pi}{i\xi^0} , \quad z^a = \frac{\xi^a}{\xi^0} .$$

between holomorphic topological amplitude and real polarized topological amplitude, and identifying $Q_a = \hat{q}_a + \frac{c_{2,a}}{24}$, $2J = \hat{q}_0$,

Schwartz Tang

Single fivebrane and topological string amplitude II

- We arrive at a relation between the NS5-brane partition function in type IIB/$\hat{\mathcal{X}}$ and the A-model topological amplitude on $\hat{\mathcal{X}}$:

$$H_{k=1}(\xi^\Lambda) = e^{-f_1(\xi^a/\xi^0)} \left(\xi^0\right)^{\frac{\chi}{24}-1} [M(e^{2\pi i/\xi^0})]^{\frac{\chi_{\hat{\mathcal{X}}}}{2}} \Psi^{\rm top}_{\mathbb{R}}(\xi^\Lambda) \,.$$

  Note that $(-1)^{2J}$ follows from quadratic refinement $\sigma(\gamma)$, we seem to find $\theta^\Lambda = 0$, $\phi_0 = A_{00}/2$, while ϕ_a remains undetermined. The prefactors are puzzling, and need further understanding.
- By mirror symmetry, the same formula should relate the NS5-brane partition function in type IIA/$\mathcal{X}$ and the B-model topological amplitude on $\mathcal{X}$, as anticipated by various authors.
- Integrating over the $\mathbb{P}^1$ fiber in the weak coupling limit, we recover the partition function of a Gaussian self-dual three-form on $\mathcal{X}$ discussed previously. The auxiliary parameter of DVV is identified as the twistor coordinate t.

Outline

1. Introduction
2. Topology of the HM moduli space
3. Qualitative aspects of five-brane corrections
4. Mirror symmetry, S-duality and (p, k) fivebranes
5. Conclusion

Conclusion I

- We have determined the topology of the HM moduli space in type IIA$/\mathcal{X}$ at fixed (weak) coupling:

$$\mathcal{M} = \mathbb{R}^+_r \times \begin{pmatrix} S^1_\sigma & \to & \mathcal{C}(r) \\ & & \downarrow \\ & & \mathcal{J}(\mathcal{X}) \end{pmatrix} ,$$

  where $\mathcal{J}(\mathcal{X})$ is the intermediate Jacobian of the CY family $\mathcal{X}$, $\mathcal{C}(r)$ is the circle bundle $\mathcal{L}_\Theta^{-1} \otimes \mathcal{L}^{-\chi_\mathcal{X}/24}$, and the characteristics $\Theta \in \{0, \frac{1}{2}\}^{b_3(\mathcal{X})}$ are extra data that must be specified.
- The same holds in type IIB$/\hat{\mathcal{X}}$ by replacing $\mathcal{J}$ by the torus bundle over $\mathcal{M}_K(\hat{\mathcal{X}})$ with fiber $H^{\rm even}(\hat{\mathcal{X}}, \mathbb{R})/K(\hat{\mathcal{X}})$, $b_3(\mathcal{X}) \to 2b_2(\hat{\mathcal{X}}) + 2$, $\chi_\mathcal{X} \to -\chi_{\hat{\mathcal{X}}}$.
- What is the topology of the full HM space ? Is the singularity at $r = \chi_\mathcal{X}/96\pi$ resolved by quantum effects ?

Conclusion II

- Euclidean D-branes correct the metric at order e^{-1/g_s}. They are mostly under control (barring convergence issues). We believe that the quadratic refinement entering in these corrections is the same $\sigma_\Theta(\gamma)$ as above.
- NS5-branes correct the metric at order e^{-1/g_s^2}. We have taken steps in computing these effects, by applying S-duality to D-brane effects. To complete the story, one should resolve subtle phase issues, fix contours, ETC.
- We found some tension between S-duality, Heisenberg invariance and monodromy invariance. Presumably S-duality is broken to a subgroup $\Gamma(N)$ where N is the common denominator of $c_{2a}/24$. It would be very interesting to study the sector with no D5/NS5-branes, which should be S-duality invariant by itself.