Hypermultiplet moduli spaces in type II string theories: a mini-survey

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based on work with Alexandrov, Saueressig, Vandoren, Persson, ...

Propaganda

- In $D = 4$ string vacua with $N = 2$ supersymmetries , the moduli space splits into a product $\mathcal{M} = VM_4 \times HM_4$ corresponding to vector multiplets and hypermultiplets.

$$\mathrm{IIA}/\mathcal{X} \;\Big|\; \mathrm{IIB}/\hat{\mathcal{X}} \;\Big|\; \mathrm{Het}/K_3 \times T^2 \;\Big|\; \dots$$

- The study of VM_4 and of the BPS spectrum has had tremendous applications in mathematics and physics: classical mirror symmetry, Gromov-Witten invariants, Donaldson-Thomas invariants, black hole precision counting, etc...
- Understanding HM_4 may be even more rewarding: quantum extension of mirror symmetry, new geometric invariants, new checks of Het/II duality richer automorphic properties...

Hypers = Vectors

- Upon circle compactification to $D = 3$,the VM and HM moduli spaces become two sides of the same coin, exchanged by T-duality along the circle.
- VM_3 includes VM_4, the electric and magnetic holonomies of the $D = 4$ Maxwell fields, the radius R of the circle and the NUT potential σ, dual to the Kaluza-Klein gauge field in $D = 3$:

$$\begin{aligned} \mathrm{VM}_3 &\approx \text{c-map}(\mathrm{VM}_4) + \text{1-loop} + \mathcal{O}(e^{-R}) + \mathcal{O}(e^{-R^2}) \\ \mathrm{HM}_3 &= \mathrm{HM}_4 \end{aligned}$$

- SUSY requires that both VM_3 and HM_3 are quaternion-Kähler manifolds.

Instantons = Black holes + KKM

- The $\mathcal{O}(e^{-R})$ corrections come from BPS black holes in $D = 4$, whose Euclidean wordline winds around the circle: thus VM_3 encodes the $D = 4$ spectrum, with chemical potentials for every electric and magnetic charges, and naturally incorporates chamber dependence.

Seiberg Witten; Shenker

- The $\mathcal{O}(e^{-R^2})$ corrections come from Kaluza-Klein monopoles, i.e. gravitational instantons of the form $\mathrm{TN}_k \times \mathcal{Y}$ ($\mathcal{Y} = \hat{\mathcal{X}}, \mathcal{X}, K_3 \times T^2$). (in Lorentzian signature, these would have closed timelike curves).
- Including these additional contributions will (hopefully) lead to enhanced automorphic properties, analogous to the $SL(2,\mathbb{Z}) \to Sp(2,\mathbb{Z})$ enhancement in $N = 4$ dyon counting.

Dijkgraaf Verlinde Verlinde; Gunaydin Neitzke BP Waldron

SYM vs. SUGRA

- A much simpler version of this problem occurs in (Seiberg-Witten) $\mathcal{N} = 2$ SYM field theories on $\mathbb{R}^3 \times S^1$. In this case VM_3 is a hyperkähler manifold of the form

$$\mathrm{VM}_3 \approx \text{rigid c-map}(\mathrm{VM}_4) + \mathcal{O}(e^{-R})$$

- The $\mathcal{O}(e^{-R})$ corrections similarly come from BPS dyons in $D = 4$. Understanding their effect on the complex symplectic structure of the twistor space $\mathcal{Z}$ of VM_3 has lead to a physical derivation of the KS wall-crossing formula.

Gaiotto Moore Neitzke, Kontsevich Soibelman

- The extension to $\mathcal{N} = 2$ SUGRA is non-trivial, due (in part) to the exponential growth of BPS degeneracies, and poor understanding of KK monopoles. In fact, KKM contributions appears to be needed in order to resolve the ambiguity of the black hole asymptotic series.

BP Vandoren

Back to HM

- On the flip side of the coin, $R = 1/g_{(4)}$ is the inverse string coupling. The $\mathcal{O}(e^{-R})$ and $\mathcal{O}(e^{-R^2})$ corrections to HM_4 now originate from Euclidean D-branes and NS5-branes, respectively.

  Becker Becker Strominger

- When $\mathcal{X}$ is K3-fibered, HM_4 can in principle be computed exactly using Het/type II duality: since the heterotic string coupling belongs to VM_4, HM_4 is determined by the $(0,4)$ heterotic SCFT at tree level (still non-trivial due to non-perturbative α' corrections)

  Aspinwall

- Recent progress has instead occurred on the type II side, combining S-duality and mirror symmetry with an improved understanding of twistor techniques.

  Robles-Llana Rocek Saueressig Theis Vandoren

  Alexandrov BP Saueressig Vandoren

Outline

Outline

The perturbative metric I

- The HM moduli space in type IIA compactified on a CY 3-fold (family) $\mathcal{X}$ is a quaternion-Kähler manifold $\mathcal{M}$ of real dimension $2b_3(\mathcal{X}) = 4(h_{2,1}+1)$.
- $\mathcal{M} \equiv \mathcal{Q}_c(\mathcal{X})$ encodes
  1. the 4D dilaton $R \equiv 1/g_{(4)}$,
  2. the complex structure of the CY family $\mathcal{X}$,
  3. the periods of the RR 3-form C on $\mathcal{X}$,
  4. the NS axion σ, dual to the Kalb-Ramond B-field in 4D
- To write down the metric explicitly, let us choose a symplectic basis $\mathcal{A}^\Lambda, \mathcal{B}_\Lambda$, $\Lambda = 0 \dots h_{2,1}$ of $H_3(\mathcal{X},\mathbb{Z})$.

The perturbative metric II

- The complex structure moduli space $\mathcal{M}_c(\mathcal{X})$ may be parametrized by the periods $\Omega(z^a) = (X^\Lambda, F_\Lambda) \in H_3(\mathcal{X}, \mathbb{C})$ of the (3,0) form

$$X^\Lambda = \int_{\mathcal{A}^\Lambda} \Omega_{3,0} \, , \quad F_\Lambda = \int_{\mathcal{B}_\Lambda} \Omega_{3,0} \, ,$$

  up to holomorphic rescalings $\Omega \mapsto e^f \Omega$.

- $\mathcal{M}_c(\mathcal{X})$ is endowed with a special Kähler metric

$$\mathrm{d}s^2_{\mathcal{SK}} = \partial\bar\partial\mathcal{K} \, , \qquad \mathcal{K} = -\log[\mathrm{i}(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda)]$$

  and a $\mathbb{C}^\times$ bundle $\mathcal{L}$ with connection $\mathcal{A}_K = \frac{\mathrm{i}}{2}(\mathcal{K}_a \mathrm{d}z^a - \mathcal{K}_{\bar{a}} \mathrm{d}\bar{z}^{\bar{a}})$.

- Ω transforms as $\Omega \mapsto e^f \rho(M)\, \Omega$ under a monodromy M in $\mathcal{M}_c(\mathcal{X})$, where $\rho(M) \in Sp(b_3, \mathbb{Z})$.

The perturbative metric III

- Topologically trivial harmonic C-fields on $\mathcal{X}$ may be parametrized by the real periods

$$\zeta^\Lambda = \int_{\mathcal{A}^\Lambda} C, \quad \tilde{\zeta}_\Lambda = \int_{\mathcal{B}_\Lambda} C\,.$$

- Large gauge transformations require that $C \equiv (\zeta^\Lambda, \tilde{\zeta}_\Lambda)$ takes values in the intermediate Jacobian torus

$$C \in T = H^3(\mathcal{X}, \mathbb{R})/H^3(\mathcal{X}, \mathbb{Z})$$

  i.e. that $(\zeta^\Lambda, \tilde{\zeta}_\Lambda)$ have unit periodicities.

- This is consistent with D-instanton charge quantization, as we shall discuss later.

The perturbative metric IV

- T carries a canonical symplectic form and complex structure induced by the Hodge $\star_{\mathcal{X}}$, hence a Kähler metric

$$\mathrm{d}s_T^2 = -\frac{1}{2}(\mathrm{d}\tilde{\zeta}_\Lambda - \bar{\mathcal{N}}_{\Lambda\Lambda'}\mathrm{d}\zeta^{\Lambda'})\mathrm{Im}\mathcal{N}^{\Lambda\Sigma}(\mathrm{d}\tilde{\zeta}_\Lambda - \mathcal{N}_{\Sigma\Sigma'}\mathrm{d}\zeta^{\Sigma'})$$

where

$$\mathcal{N}_{\Lambda\Lambda'} = \bar{\tau}_{\Lambda\Lambda'} + 2\mathrm{i}\frac{[\mathrm{Im}\tau\cdot X]_\Lambda[\mathrm{Im}\tau\cdot X]_{\Lambda'}}{X^\Sigma\,\mathrm{Im}\tau_{\Sigma\Sigma'}X^{\Sigma'}}\,, \qquad \tau_{\Lambda\Sigma} = \partial_{X^\Lambda}\partial_{X^\Sigma}F$$

- $\mathcal{N}$ (resp. τ) is the Weil (resp. Griffiths) period matrix of $\mathcal{X}$. While $\mathrm{Im}\tau$ has signature $(1, b_3 - 1)$, $\mathrm{Im}\mathcal{N}$ is negative definite.
- Under monodromies, $C \mapsto \rho(M)C$. We shall refer to the total space of the torus bundle $T \to \mathcal{J}_c(\mathcal{X}) \to \mathcal{M}_c(\mathcal{X})$ as the (Weil) intermediate Jacobian of $\mathcal{X}$.

The tree-level metric

- At tree level, i.e. in the strict weak coupling limit $R = \infty$, the quaternion-Kähler metric on $\mathcal{M}$ is given by the *c*-map metric

$$ds^2_{\mathcal{M}} = \frac{4}{R^2}\,\mathrm{d}R^2 + 4\,\mathrm{d}s^2_{\mathcal{SK}} + \frac{\mathrm{d}s^2_T}{R^2} + \frac{1}{16R^4} D\sigma^2\,.$$

where

$$D\sigma \equiv \mathrm{d}\sigma + \langle C, \mathrm{d}C\rangle = \mathrm{d}\sigma + \tilde{\zeta}_\Lambda \mathrm{d}\zeta^\Lambda - \zeta^\Lambda \mathrm{d}\tilde{\zeta}_\Lambda$$

Cecotti Girardello Ferrara; Ferrara Sabharwal

- The *c*-map metric admits continuous isometries

$$T_{H,\kappa} : (C, \sigma) \mapsto (C + H, \sigma + 2\kappa + \langle C, H\rangle)$$

where $H \in H^3(\mathcal{X}, \mathbb{R})$ and $\kappa \in \mathbb{R}$, satisfying the Heisenberg group relation

$$T_{H_1,\kappa_1} T_{H_2,\kappa_2} = T_{H_1+H_2,\kappa_1+\kappa_2+\frac{1}{2}\langle H_1,H_2\rangle}\,.$$

The one-loop corrected metric I

- The one-loop correction deforms the metric on $\mathcal{M}$ into

$$ds^2_{\mathcal{M}} = 4\frac{R^2+2c}{R^2(R^2+c)}\,\mathrm{d}R^2 + \frac{4(R^2+c)}{R^2}\,\mathrm{d}s^2_{\mathcal{SK}} + \frac{\mathrm{d}s^2_T}{R^2} + \frac{2\,c}{R^4}\,e^{\mathcal{K}}\,|X^\Lambda \mathrm{d}\tilde{\zeta}_\Lambda - F_\Lambda \mathrm{d}\zeta^\Lambda|^2 + \frac{R^2+c}{16R^4(R^2+2c)} D\sigma^2 \,.$$

where $D\sigma = \mathrm{d}\sigma + \langle C, \mathrm{d}C\rangle + 8c\mathcal{A}_K\,, \quad c = -\chi(\mathcal{X})/(192\pi)$

Antoniadis Minasian Theisen Vanhove; Gunther Hermann Louis; Robles-Llana Saueressig Vandoren

- The one-loop correction to g_{rr} was computed by reducing the CP-even R^4 coupling in 10D on $\mathcal{X}$. The correction to $D\sigma$ can be obtained with less effort by reducing CP-odd couplings in 10D.

The one-loop corrected metric II

- Consider the topological coupling in $D = 10$ type IIA supergravity:

$$\int_{\mathcal{Y}} \left(\frac{1}{6} B \wedge \mathrm{d}C \wedge \mathrm{d}C - B \wedge I_8 \right), \quad I_8 = \frac{1}{48}(p_2 - \frac{1}{4} p_1^2)$$

- On a complex 10-manifold,

$$B \wedge I_8 = \frac{1}{24} B \wedge \left[c_4 - c_1 \left(c_3 + \frac{1}{8} c_1^3 - \frac{1}{2} c_1 c_2 \right) \right].$$

- Integrating on $\mathcal{X}$ and using $c_4 = 0, c_3 = \chi(\mathcal{X}), c_1 = -\omega_c$ leads to

$$\int d^4x \left[\mathrm{Re}\mathcal{N}_{\Lambda\Sigma} (\mathrm{d}C^\Lambda + \zeta^\Lambda \mathrm{d}B) \wedge d\zeta^\Sigma - \frac{\chi(\mathcal{X})}{24\pi} B \wedge \omega_c \right]$$

where $C^\Lambda = \int_{\mathcal{A}^\Lambda} C$. Dualizing the two-forms C^Λ, B into $\tilde{\zeta}_\Lambda, \sigma$ produces the one-form $D\sigma$ indicated previously.

The one-loop corrected metric III

- The one-loop correction to $D\sigma$ has important implications for the topology of the HM moduli space, as we shall discuss later.
- The one-loop corrected metric is presumably exact to all orders in $1/R$. It will receive $\mathcal{O}(e^{-R})$ and $\mathcal{O}(e^{-R^2})$ corrections from D-instantons and NS5-brane instantons, eventually breaking all continuous isometries.
- Note the curvature singularity at finite distance $R^2 = -2c$ when $\chi(\mathcal{X}) > 0$! This should hopefully be resolved by instanton corrections.

Outline

Topology of the HM moduli space I

- At least at weak coupling, $\mathcal{M}$ is foliated by hypersurfaces $\mathcal{C}(R)$ of constant string coupling. We shall now discuss the topology of the leaves $\mathcal{C}(R)$.
- Quotienting by translations along the NS axion σ, we already saw that $\mathcal{C}(R)/\partial_\sigma$ reduces to the intermediate Jacobian $\mathcal{J}_c(\mathcal{X})$, in particular $C \in T = H^3(\mathcal{X},\mathbb{R})/H^3(\mathcal{X},\mathbb{Z})$.
- This is consistent with the fact that Euclidean D2-branes wrapping a special Lagrangian submanifold in integer homology class $\gamma = q_\Lambda \mathcal{A}^\Lambda - p^\Lambda \mathcal{B}_\Lambda \in H_3(\mathcal{X},\mathbb{Z})$ induce corrections roughly of the form

$$\delta \mathrm{d}s^2|_{D2} \sim \exp\left(-8\pi \frac{|Z_\gamma|}{g_{(4)}} - 2\pi \mathrm{i}\langle\gamma, C\rangle\right) .$$

Here $Z_\gamma \equiv e^{\mathcal{K}/2}(q_\Lambda X^\Lambda - p^\Lambda F_\Lambda)$ is the central charge.

Topology of the HM moduli space II

- More precisely, but still schematically,

$$\delta \mathrm{d}s^2|_{\mathrm{D2}} \sim \sigma_D(\gamma)\,\bar\Omega(\gamma, z^a)\,\exp\left(-8\pi\frac{|Z_\gamma|}{g_{(4)}} - 2\pi\mathrm{i}\langle\gamma, C\rangle\right)\,.$$

  where $\bar\Omega(\gamma, z^a)$ is the (generalized) Donaldson-Thomas invariant associated to γ with stability condition depending on z^a, and $\sigma_D : H_3(\mathcal{X}, \mathbb{Z}) \to U(1)$ is a quadratic refinement of the symplectic pairing,

$$\sigma_D(\gamma+\gamma') = (-1)^{\langle\gamma,\gamma'\rangle}\,\sigma_D(\gamma)\,\sigma_D(\gamma')\,.$$

  The choice of σ_D amounts to a choice of characteristics, as we explain momentarily.
- The exact form of the D2-instanton corrections is dictated by wall-crossing in twistor space, as we shall discuss later. It would be interesting to derive the prefactor from a one-loop determinant.

Gaiotto Moore Neitzke; Harvey Moore

Topology of the HM moduli space III

- NS5-brane instantons will further break continuous translations along σ to discrete shifts $\sigma \mapsto \sigma + 2$ (in our conventions). Thus $\mathcal{C}(R)$ is a circle bundle over $\mathcal{J}_c(\mathcal{X})$, with fiber parametrized by $e^{\mathrm{i}\pi\sigma}$.
- The horizontal one-form $D\sigma = \mathrm{d}\sigma + \langle C, \mathrm{d}C \rangle - \frac{\chi(\mathcal{X})}{24\pi} \mathcal{A}_K$ implies that the first Chern class of $\mathcal{C}$ is

$$\mathrm{d}\left(\frac{D\sigma}{2}\right) = \omega_T + \frac{\chi(\mathcal{X})}{24}\,\omega_c \,, \quad \omega_T = \mathrm{d}\tilde{\zeta}_\Lambda \wedge \mathrm{d}\zeta^\Lambda \,, \quad \omega_c = -\frac{1}{2\pi}\mathrm{d}\mathcal{A}_K$$

  where ω_T, ω_c are the Kähler forms on T and $\mathcal{M}_c(\mathcal{X})$.
- The first term means that large gauge transformations $C \to C + H$ commute up to a shift of σ. The second term means that σ also shifts under monodromies in $\mathcal{M}_c(\mathcal{X})$. To determine these shifts, let us examine NS5-instanton corrections.

Five-brane instantons I

- NS5-brane instantons with charge $k \in \mathbb{Z}$ are expected to produce corrections to the metric of the form

$$\delta \mathrm{d}s^2|_{\mathrm{NS5}} \sim \exp\left(-4\pi|k|/g^2_{(4)} - \mathrm{i}k\pi\sigma\right)\, \mathcal{Z}^{(k)}(z^a, C)\ ,$$

  where $\mathcal{Z}^{(k)} = \mathrm{Tr}[(2J_3)^2(-1)^{2J_3}]$ is the (twisted) partition function of the world-volume theory on a stack of k five-branes. For this to be globally well-defined, $\mathcal{Z}^{(k)}$ must be a section of $[\mathcal{C}(R)]^k$.

- Recall that the type IIA NS5-brane supports a self-dual 3-form flux $H = \mathrm{i} \star H$, together with its SUSY partners. The partition function of a self-dual form is known to be a holomorphic section of a non-trivial line bundle $\mathcal{L}^k_{\mathrm{NS5}}$ over the space of metrics and C fields.

Witten; Henningson Nilsson Salomonson; Belov Moore; . . .

Five-brane instantons II

- Indeed, the restriction $\mathcal{L}_{\rm NS5}|_T$ is known to be a line bundle with first Chern class $c_1 = \omega_T$. To specify this bundle, one must choose holonomies $\sigma(H) \in U(1)$ around each cycle $H \in H_3(\mathcal{X}, \mathbb{Z})$, such that

$$\sigma(H + H') = (-1)^{\langle H, H' \rangle}\, \sigma(H)\, \sigma(H')\,.$$

  S-duality suggests that $\sigma = PD[\sigma_D]$, but this needs to be clarified.

- The general solution can be parametrized by characteristics $\Theta \in H_3(\mathcal{X}, \mathbb{R})/H_3(\mathcal{X}, \mathbb{Z})$ (notation: $\mathrm{E}^x \equiv e^{2\pi \mathrm{i} x}$)

$$\sigma(H) = \mathrm{E}^{-\frac{1}{2} n^\Lambda m_\Lambda + \langle H, \Theta \rangle}\,, \quad H = (n^\Lambda, m_\Lambda)\,, \quad \Theta = (\theta^\Lambda, \phi_\Lambda)$$

Five-brane instantons III

- The bundle $(\mathcal{L}_\Theta)^k$ is then defined by the twisted periodicity condition
$$\mathcal{Z}(\mathcal{N}, C+H) = \sigma^k(H)\, \mathrm{E}^{\frac{k}{2}\langle H,C\rangle} \mathcal{Z}(\mathcal{N}, C)$$
- Holomorphic sections of $(\mathcal{L}_\Theta)^k$ are Siegel theta series of rank $b_3(\mathcal{X})$, level $k/2$,
$$\mathcal{Z}^{(k)}_\mu(\mathcal{N}, C) = N \sum_{n^\Lambda \in \Gamma_m + \mu + \theta} \mathrm{E}^{\frac{k}{2}(\zeta^\Lambda - n^\Lambda)\bar{\mathcal{N}}_{\Lambda\Sigma}(\zeta^\Sigma - n^\Sigma) + k(\tilde\zeta_\Lambda - \phi_\Lambda)n^\Lambda + \frac{k}{2}(\theta^\Lambda \phi_\Lambda - \zeta^\Lambda \tilde\zeta_\Lambda)},$$
where Γ_m is a Lagrangian sublattice of $\Gamma = H^3(\mathcal{X}, \mathbb{Z})$, and μ runs over $(\Gamma_m/k)/\Gamma_m$.
- This agrees with the chiral five-brane partition function obtained by holomorphic factorization of the partition function of a non-chiral 3-form $H = \mathrm{d}\mathcal{B}$ on $\mathcal{X}$, with Gaussian action. The C-independent normalization factor N is tricky.

Topology of the NS axion I

- For the coupling $e^{-\mathrm{i}\pi k\sigma}\mathcal{Z}^{(k)}$ to be invariant under large gauge transformations, $e^{\mathrm{i}\pi\sigma}$ must also transform as a section of $\mathcal{L}_\Theta$. Therefore, σ must pick up additional shifts under discrete translations along T,

$$T'_{H,\kappa} : (C,\sigma) \mapsto \left(C + H, \sigma + 2\kappa + \langle C, H\rangle - n^\Lambda m_\Lambda + 2\langle H, \Theta\rangle\right)$$

  where $H \equiv (n^\Lambda, m_\Lambda) \in \mathbb{Z}^{b_3}$, $\kappa \in \mathbb{Z}$. This is needed for the consistency of large gauge transformations,

$$T'_{H_1,\kappa_1} T'_{H_2,\kappa_2} = T'_{H_1+H_2,\kappa_1+\kappa_2+\frac{1}{2}\langle H_1,H_2\rangle + \frac{1}{2\pi\mathrm{i}}\log\frac{\sigma(H_1+H_2)}{\sigma(H_1)\sigma(H_2)}}$$

- The transformation of $e^{\mathrm{i}\pi\sigma}$ under monodromies in $\mathcal{M}_c(\mathcal{X})$ must also cancel that of $\mathcal{Z}^{(k)}_\mu$. This is guaranteed by the anomaly inflow mechanism, but details remain to be worked out.

Outline

A lightning review of twistors I

- QK manifolds $\mathcal{M}$ are conveniently described via their twistor space $\mathbb{P}^1 \to \mathcal{Z} \to \mathcal{M}$, a complex contact manifold with real involution. Choosing a stereographic coordinate t on $\mathbb{P}^1$, the contact structure is the kernel of the local (1,0)-form

$$Dt = \mathrm{d}t + p_+ - \mathrm{i}p_3 t + p_- t^2$$

where $p_3, p_\pm$ are the $SU(2)$ components of the Levi-Civita connection on $\mathcal{M}$. Dt is well-defined modulo rescalings.

- $\mathcal{Z}$ is further equipped with a Kähler-Einstein metric

$$\mathrm{d}s^2_{\mathcal{Z}} = \frac{|Dt|^2}{(1+t\bar{t})^2} + \frac{\nu}{4}\mathrm{d}s^2_{\mathcal{M}} \,, \qquad \nu = \frac{R(\mathcal{M})}{4d(d+2)}$$

If $\mathcal{M}$ has negative scalar curvature, $\mathcal{Z}$ is pseudo-Kähler with signature $(2, \dim \mathcal{M})$.

see Salamon's lecture

A lightning review of twistors II

- Rk: complex contact manifolds are projectivizations of complex symplectic cones. The $\mathbb{C}^\times$ bundle over $\mathcal{Z}$ is the hyperkähler cone associated to $\mathcal{M}$. The two approaches are essentially equivalent.

Swann; de Wit Rocek Vandoren

- Locally, there always exist Darboux coordinates $(\Xi, \tilde\alpha) = (\xi^\Lambda, \tilde\xi_\Lambda, \tilde\alpha)$ and a "contact potential" Φ such that

$$2\, e^\Phi \frac{Dt}{\mathrm{i}t} = \mathrm{d}\tilde\alpha + \langle \Xi, \mathrm{d}\Xi \rangle = \mathrm{d}\tilde\alpha + \tilde\xi_\Lambda \mathrm{d}\xi^\Lambda - \xi^\Lambda \mathrm{d}\tilde\xi_\Lambda\ .$$

- The contact potential is holomorphic on $\mathbb{P}^1$, and provides a Kähler potential for the Kähler metric on $\mathcal{Z}$ via $e^{K_\mathcal{Z}} = (1 + t\bar t) e^{Re(\Phi)}/|t|$.

Alexandrov BP Saueressig Vandoren

A lightning review of twistors III

- By the moment map construction, continuous isometries of $\mathcal{M}$ are in 1-1 correspondence with classes in $H^0(\mathcal{Z}, \mathcal{O}(2))$. In particular, any continuous isometry of $\mathcal{M}$ can be lifted to a holomorphic action on $\mathcal{Z}$.

Salamon; Galicki Salamon

- Infinitesimal deformations of $\mathcal{M}$ lift to deformations of the complex contact transformations between Darboux coordinate patches on $\mathcal{Z}$, hence are classified by $H^1(\mathcal{Z}, \mathcal{O}(2))$.

Lebrun; Alexandrov BP Saueressig Vandoren

Twistor description of the perturbative metric

- For the one-loop corrected HM metric, the following Darboux coordinates do the job, away from the north and south poles $t = 0, \infty$:

$$\Xi = C + 2(R^2 + c)\, e^{\mathcal{K}/2} \left[t^{-1}\Omega - t\,\bar{\Omega} \right]$$
$$\tilde{\alpha} = \sigma + 2(R^2 + c)\, e^{\mathcal{K}/2} \left[t^{-1}\langle \Omega, C\rangle - t\,\langle \bar{\Omega}, C\rangle \right] - 8\mathrm{i} c \log t$$

Neitzke BP Vandoren; Alexandrov

- The isometry $T_{H,\kappa}$ acts holomorphically on $\mathcal{Z}$ by

$$(\Xi, \tilde{\alpha}) \mapsto (\Xi + H, \tilde{\alpha} + 2\kappa + \langle \Xi, H\rangle)$$

- Modding out by large gauge transformations $T'_{H,\kappa}$, $\mathcal{Z}$ becomes a complexified twisted torus $\mathbb{C}^\times \ltimes [H^3(\mathcal{X}, \mathbb{Z}) \otimes \mathbb{C}^\times]$.

D-instantons in twistor space I

- D-instanton corrections to $\mathcal{Z}$ are essentially dictated by wall crossing. Recall that the Kontsevich-Soibelman wall-crossing formula requires that the product

$$\prod_\gamma U_\gamma \, , \qquad U_\gamma \equiv \exp\left(\Omega(\gamma; t^a) \sum_{d=1}^{\infty} \frac{e_{d\gamma}}{d^2} \right) \, ,$$

ordered such that $\arg(Z_\gamma)$ decreases from left to right, stays invariant across the wall. Here $\Omega(\gamma; t^a)$ are generalized DT invariants, and e_γ satisfy the Lie algebra

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \, e_{\gamma_1+\gamma_2} \, .$$

D-instantons in twistor space II

- Defining $\delta_\gamma = \sigma(\gamma) e_\gamma$, we see that we can represent δ_γ a a Hamiltonian vector field on Z,

$$\delta_\gamma = (\partial_{\xi^\Lambda} \mathcal{X}_\gamma)\, \partial_{\tilde\xi_\Lambda} - (\partial_{\tilde\xi_\Lambda} \mathcal{X}_\gamma)\, \partial_{\xi^\Lambda} + 2\mathrm{i}[(2 - \xi^\Lambda \partial_{\xi^\Lambda} - \tilde\xi_\Lambda \partial_{\tilde\xi_\Lambda}) \mathcal{X}_\gamma]\, \partial_{\tilde\alpha}$$

  where

$$\mathcal{X}_\gamma = \exp \langle \Xi, \gamma \rangle = \mathrm{E}^{q_\Lambda \xi^\Lambda - p^\Lambda \tilde\xi_\Lambda}$$

- Exponentiating, U_γ implements the contact transformation

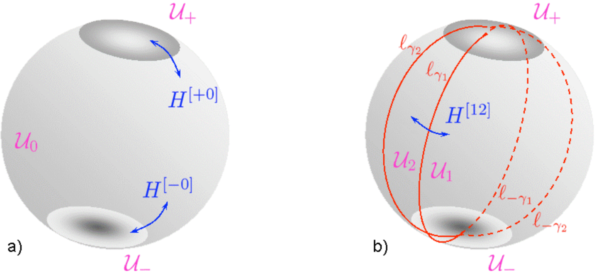
$$\mathcal{X}_{\gamma'} \mapsto \mathcal{X}_{\gamma'} (1 - \sigma(\gamma) \mathcal{X}_\gamma)^{\langle \gamma, \gamma' \rangle\, \Omega(\gamma)} , \qquad \tilde\alpha \mapsto \tilde\alpha - \frac{1}{2\pi^2} \Omega(\gamma)\, L[\sigma(\gamma) \mathcal{X}_\gamma]$$

  where $L(z) = \sum_{i=1}^{\infty} \frac{z^n}{n^2} + \frac{1}{2} \log z \, \log(1 - z)$ is Rogers' dilogarithm.

- The projection to the complexified torus $H^3(\mathcal{X}, \mathbb{Z}) \otimes \mathbb{C}^\times$ reduces to the symplectomorphism considered by GMN in the context of HK geometry / gauge theories.

D-instantons in twistor space III

- By analogy with GMN, it is natural to propose that the D-instanton corrected twistor space is obtained by gluing Darboux coordinate patches along BPS rays $\ell_\pm = \{t \ : \ Z(\gamma; z^a)/t \in \pm i\mathbb{R}^+\}$, using the contact transformation U_γ. The consistency of this prescription across lines of marginal stability is guaranteed by the KS formula.



- This can also be (and was first) argued from type IIB S-duality.

D-instantons in twistor space IV

- These gluing conditions for $\Xi = (\xi^\Lambda, \tilde{\xi}_\Lambda)$ can be summarized by integral equations

$$\Xi = \Xi_{\rm sf} - \frac{1}{8\pi^2} \sum_{\gamma'} \Omega(\gamma') \langle \gamma, \gamma' \rangle \int_{l_{\gamma'}} \frac{\mathrm{d}t'}{t'} \frac{t+t'}{t-t'} \mathrm{Li}_1 \left[\sigma_{\rm D}(\gamma') \, \mathrm{E}^{-\langle \Xi(t'), \gamma' \rangle} \right],$$

  where $\mathrm{Li}_1(x) \equiv -\log(1-x)$. These are formally identical to Zamolodchikov's Y-system in studies of integrable models.

  GMN; Alexandrov Roche

- These eqs can be solved iteratively, by first substituting $\Xi \to \Xi_{\rm sf}$ on the rhs, integrating, etc. leading to an infinite series of multi-instanton corrections.
- Having determined Ξ in each patch, one can then compute $\tilde{\alpha}$, Φ, and extract the QK metric as an asymptotic series. The series has zero radius of convergence, due to (presumed) exponential growth of DT invariants.

D-instantons in twistor space V

- E.g. in the one-instanton approximation, the contact potential is given by

$$e^{\Phi} = e^{\Phi_{\rm sf}} + \frac{1}{4\pi^2} \sum_{\gamma \in \Gamma} \sigma_{\rm D}(\gamma)\, \bar{\Omega}(\gamma) K_1(4\pi |Z(\gamma)|/g_4)\, \cos\left(2\pi \langle C, \gamma \rangle\right) + \dots$$

  where $\bar{\Omega}(\gamma)$ are the rational DT invariants

$$\bar{\Omega}(\gamma) = \sum_{d|\gamma} \frac{1}{d^2}\, \Omega(\gamma/d)\,.$$

- By construction, the metric is smooth across walls of marginal stability, with one-instanton effects on one side being traded for multi-instanton effects on the other side.

Outline

HM moduli space in type IIB I

- The HM moduli space in type IIB compactified on a CY 3-fold $\hat{\mathcal{X}}$ is a QK manifold $\mathcal{M} \equiv \mathcal{Q}_K(\hat{\mathcal{X}})$ of real dimension $4(h_{1,1}+1)$
  1. the 4D dilaton $R \equiv 1/g_{(4)}$,
  2. the complexified Kähler moduli $z^a = b^a + \mathrm{i} t^a = X^a/X^0$
  3. the periods of $C = C^{(0)} + C^{(2)} + C^{(4)} + C^{(6)} \in H^{\mathrm{even}}(\hat{\mathcal{X}}, \mathbb{R})$
  4. the NS axion σ
- Near the infinite volume point, $\mathcal{M}_K(\hat{\mathcal{X}})$ is governed by

$$F(X) = -\frac{N(X^a)}{X^0} + \frac{1}{2} A_{\Lambda\Sigma} X^\Lambda X^\Sigma + \chi(\hat{\mathcal{X}}) \frac{\zeta(3)(X^0)^2}{2(2\pi \mathrm{i})^3} + F_{\mathrm{GW}}(X)$$

where $N(X^a) \equiv \frac{1}{6}\kappa_{abc} X^a X^b X^c$, κ_{abc} is the cubic intersection form, $A_{\Lambda\Sigma}$ is a constant, real symmetric matrix, defined up to integer shifts and F_{GW} are Gromov-Witten instanton corrections:

$$F_{\mathrm{GW}}(X) = -\frac{(X^0)^2}{(2\pi \mathrm{i})^3} \sum_{k_a \gamma^a \in H_2^+(\hat{\mathcal{X}})} n^{(0)}_{k_a} \, \mathrm{Li}_3 \left[\mathrm{E}^{k_a \frac{X^a}{X^0}} \right] ,$$

HM moduli space in type IIB II

- Quantum mirror symmetry implies $\mathcal{Q}_c(\mathcal{X}) = \mathcal{Q}_K(\hat{\mathcal{X}})$. At the perturbative level, this reduces to classical mirror symmetry.
- D-instantons are now Euclidean D5-D3-D1-D(-1), described mathematically by coherent sheaves E on $\mathcal{X}$. Their charge vector γ is related to the Chern classes via the Mukai map

$$q_\Lambda X^\Lambda - p^\Lambda F_\Lambda = \int_{\hat{\mathcal{X}}} e^{-(B+\mathrm{i}J)}\, \mathrm{ch}(E)\, \sqrt{\mathrm{Td}(\hat{\mathcal{X}})}$$

- Assuming that $A_{\Lambda\Sigma}$ satisfies the congruences

$$A_{00} \in \mathbb{Z}\,, \quad A_{0a} \in \frac{c_{2,a}}{24} + \mathbb{Z}\,, \quad \frac{1}{2}\kappa_{abc}p^b p^c - A_{ab}p^b \in \mathbb{Z} \quad \text{for } \forall p^a \in \mathbb{Z}\,,$$

  the D-instanton charge vector $\gamma \in H^{\text{even}}(\hat{\mathcal{X}}, \mathbb{Z})$, hence C takes values in the symplectic Jacobian $T = H^{\text{even}}(\hat{\mathcal{X}}, \mathbb{R})/H^{\text{even}}(\hat{\mathcal{X}}, \mathbb{Z})$.

HM moduli space in type IIB III

- It is often convenient to eliminate $A_{\Lambda\Sigma}$ by a non-integer symplectic transformation, leading to non-integer electric charges q'_Λ,

$$q'_\Lambda = q_\Lambda - A_{\Lambda\Sigma} p^\Sigma \,, \quad \tilde\zeta'_\Lambda = \tilde\zeta_\Lambda - A_{\Lambda\Sigma}\zeta^\Lambda \,, \quad F' = F - \frac{1}{2} A_{\Lambda\Sigma} X^\Lambda X^\Sigma$$

$$q'_a \in \mathbb{Z} - \frac{p^0}{24} c_{2,a} - \frac{1}{2}\kappa_{abc} p^b p^c \,, \qquad q'_0 \in \mathbb{Z} - \frac{1}{24} p^a c_{2,a} \,,$$

- The exact HM metric should admit an isometric action of $SL(2,\mathbb{Z})$, corresponding to type IIB S-duality in 10 dimensions. This action is most easily described in the “primed" frame.

S-duality in twistor space I

- At tree level, an element $\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ acts holomorphically on $\mathcal{Z}$ via

$$\xi^0 \mapsto \frac{a\xi^0 + b}{c\xi^0 + d}, \qquad \xi^a \mapsto \frac{\xi^a}{c\xi^0 + d},$$
$$\tilde\xi'_a \mapsto \tilde\xi'_a + \frac{c}{2(c\xi^0 + d)} \kappa_{abc}\xi^b\xi^c - c_{2,a}\,\varepsilon(\delta),$$
$$\begin{pmatrix} \tilde\xi'_0 \\ \alpha' \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \tilde\xi'_0 \\ \alpha' \end{pmatrix} + \frac{1}{6}\kappa_{abc}\xi^a\xi^b\xi^c \begin{pmatrix} c^2/(c\xi^0 + d) \\ -[c^2(a\xi^0 + b) + 2c]/(c\xi^0 + d)^2 \end{pmatrix}.$$

where $\alpha' = (\tilde\alpha + \xi^\Lambda \tilde\xi'_\Lambda)/(4\mathrm{i})$. Here $\epsilon(\delta)$ is the multiplier system of the Dedekind eta function, whose necessity will become apparent later:

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right)/\eta(\tau) = \mathrm{E}^{\epsilon(\delta)}(c\tau + d)^{1/2}.$$

S-duality in twistor space II

- Continuous S-duality is broken by Gromov-Witten instantons at tree level and by the one-loop correction. A discrete subgroup can be preserved provided D(-1) and D1-instantons combine with the GW instantons into Kronecker-Eisenstein series:

$$\tau_2^{3/2}\,\mathrm{Li}_3(e^{2\pi \mathrm{i} q_a z^a}) \to \sum_{m,n}{}' \frac{\tau_2^{3/2}}{|m\tau+n|^3} e^{-S_{m,n,q}} \,,$$

where $S_{m,n,q} = 2\pi q_a |m\tau+n|\, t^a - 2\pi \mathrm{i} q_a (mc^a + nb^a)$ is the action of an (m,n)-string wrapped on $q_a\gamma^a$.

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- After Poisson resummation on $n \to q_0$, we recover the sum over D(-1)-D1 bound states, with $\Omega(0,0,q_a,q_0) = n^{(0)}_{q_a}$, $\Omega(0,0,0,0) = -\chi(\hat{\mathcal{X}})$. In particular, Li_3 turns into elliptic Li_2 !

S-duality in twistor space III

- In the presence of D3-branes, S-duality requires that the sum over D3-D1-D(-1) instantons should be a multi-variable Jacobi form of index $m_{ab} = \frac{1}{2}\kappa_{abc}p^c$ and multiplier system $\mathrm{E}^{-c_{2a}p^a\epsilon(\delta)}$.

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- The trouble is that m_{ab} has indefinite signature $(1, b_2(\hat{\mathcal{X}}) - 1)$, and the dimension of the space H^0 of such Jacobi forms vanishes. H^1 however is non-zero, and is probably where the D3-D1-D(-1) partition sum lives. This is presumably related to Mock modular forms, but details remain to be worked out.
- S-duality relates D5 and NS5. Starting from the known form of D5-D3-D1-D(-1) corrections, one may construct a Poincaré-type series to obtain the contributions from k five branes in one-instanton approximaion. This leads to a non-Gaussian generalization of the Siegel theta series based on the topological string amplitude...

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Outline

Conclusion I

- We have clarified the topology of the HM moduli space in type II/CY string vacua: the hypersurface $\mathcal{C}(r)$ at fixed (weak) coupling is a circle bundle over the Weil intermediate Jacobian in type IIA/$\mathcal{X}$, or over the “symplectic Jacobian" in type IIB/$\hat{\mathcal{X}}$. The topology of $\mathcal{C}(r)$ over the basis of the Jacobian remains to be fully determined.
- D-instanton corrections are most easily described in twistor space, and are essentially dictated by wall-crossing. The structure is a simple extension of the GMN construction to contact geometry. The divergence of the D-instanton series suggests that it may be cured by NS5-brane instanton corrections.

Conclusion II

- S-duality and mirror symmetry put powerful constraints on D-instantons and NS5-instantons. We have a rather good understanding in the one-instanton approximation, but consistency of D-instantons with S-duality / NS5-instantons with wall-crossing remained to be elucidated.
- Eventually, constraints of monodromy invariance, wall-crossing, S-duality, mirror symmetry may allow to determine the exact HM metric, at least in special cases.
- Hypers=Vectors, Instantons=Black Holes, KK monopoles = NS5-branes, join the fun with hypers !