Hypermultiplet moduli spaces in type II string theories: a survey

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based on work with Alexandrov, Saueressig, Vandoren, Persson, Manschot,

reviewed in 1304.0766

Introduction

- In $D = 4$ type II string vacua with $N = 2$ supersymmetries, the moduli space splits into a product $\mathcal{M} = \mathcal{SK} \times \mathcal{QK}$ of a special Kähler manifold, parametrized by vector multiplets and a quaternion-Kähler manifold, parametrized by hypermultiplets.
- The study of $\mathcal{SK}$ and the associated spectrum of BPS states has had many applications in mathematics and physics: classical mirror symmetry, Gromov-Witten invariants, Donaldson-Thomas invariants, black hole precision counting, etc...
- Understanding $\mathcal{QK}$ may be even more rewarding: quantum extension of mirror symmetry, new geometric invariants, new checks of dualities, richer automorphic properties...

VM multiplet moduli space in $D = 3$...

- Upon reduction to $D = 3$ on a circle $S^1(R)$, the VM moduli space $\mathcal{SK}$ extends to a quaternion-Kähler space $\widehat{\mathcal{QK}}$, which includes the electric and magnetic holonomies of the $D = 4$ Maxwell fields, the radius $R$and the NUT potential σ, dual to the KK gauge field..
- At large radius the metric on $\widehat{\mathcal{QK}}$ is given by the (one-loop deformed) c-map construction.

Cecotti Ferrara Girardello; Robles-Llana, Saueressig, Vandoren

- In addition there are $\mathcal{O}(e^{-R})$ corrections from 4D BPS states winding around the circle (e.g. D6-D2-D4-D0 branes in IIA), and $\mathcal{O}(e^{-R^2})$ corrections from gravitational instantons (Taub-NUT).

Seiberg Witten; Shenker; Gaiotto Moore Neitzke

- The metric on $\widehat{\mathcal{QK}}$ provides a kind of (tensor valued) thermal, grand-canonical BPS state partition function.

... vs. HM multiplet moduli space in $D = 4$

- The HM moduli space $\mathcal{QK}$ is unaffected by circle compactification. Moreover T-duality along the circle exchanges

$$\mathcal{QK} \leftrightarrow \widehat{\mathcal{QK}} \qquad R \leftrightarrow 1/g_4 \qquad IIA \leftrightarrow IIB$$

- The $\mathcal{O}(e^{-1/g_4})$ and $\mathcal{O}(e^{-1/g_4^2})$ corrections now arise from D5-D3-D1-D(-1) brane instantons and NS5-branes, respectively.

Becker Becker Strominger

- In both cases, D-instanton corrections depend on the generalized Donaldson-Thomas invariants, and are essentially dictated by consistency with wall crossing. Gravitational instantons or NS5-brane instantons are yet to be understood.

Alexandrov BP Saueressig Vandoren 2008

Dualities

- In type IIA on a CY threefold X, $\mathcal{QK}[X]$ and $\widehat{\mathcal{QK}}[X]$ are quaternionic versions of the complex structure and Kähler moduli spaces of X. The situation is reversed on the IIB side.
- Mirror symmetry requires $\mathcal{QK}[X] = \widehat{\mathcal{QK}}[\hat{X}]$ if $(X, \hat{X})$ are a dual pair. This includes classical mirror symmetry but goes far beyond !
- S-duality of type IIB string theory (or diffeo invariance of M-theory on $X \times T^2$) requires that $\mathcal{QK}$ and $\widehat{\mathcal{QK}}$ admits an isometric action of $SL(2,\mathbb{Z})$, constraining possible D-instanton and NS5-instantons.
- Since S-duality commutes with the large volume limit, it should hold at any level in the hierarchy

$$\begin{pmatrix} \text{one}-\text{loop} \\ D(-1) \end{pmatrix} > \begin{pmatrix} F1 \\ D1 \end{pmatrix} > D3 > \begin{pmatrix} D5 \\ NS5 \end{pmatrix}$$

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$$\begin{pmatrix} \text{one} - \text{loop} \\ D0 \end{pmatrix} > \begin{pmatrix} F1 \\ D2 \end{pmatrix} > D4 > \begin{pmatrix} D6 \\ KKM \end{pmatrix}$$

Outline

Outline

The perturbative metric I

- The HM moduli space in type IIA compactified on a CY 3-fold (family) X is a quaternion-Kähler manifold $\mathcal{M}$ of real dimension $2b_3(X) = 4(h_{2,1}(X) + 1)$.
- $\mathcal{M} \equiv \mathcal{QK}[X]$ encodes
  1. the 4D dilaton $R \equiv 1/g_{(4)}$,
  2. the complex structure of X,
  3. the periods of the RR 3-form C on X,
  4. the NS axion σ, dual to the Kalb-Ramond B-field in 4D
- To write down the metric explicitly, let us choose a symplectic basis $\mathcal{A}^\Lambda, \mathcal{B}_\Lambda$, $\Lambda = 0 \ldots h_{2,1}(X)$ of $H_3(X, \mathbb{Z})$.

The perturbative metric II

- The complex structure moduli space $\mathcal{SK}_c(X)$ may be parametrized by the periods $\Omega(z^a) = (X^\Lambda, F_\Lambda) \in H_3(X, \mathbb{C})$ of the (3,0) form

$$X^\Lambda = \int_{\mathcal{A}^\Lambda} \Omega_{3,0} \,, \quad F_\Lambda = \int_{\mathcal{B}_\Lambda} \Omega_{3,0} \,,$$

up to holomorphic rescalings $\Omega \mapsto e^f \Omega$.

- $\mathcal{M}_c(\mathcal{X})$ is endowed with a special Kähler metric

$$\mathrm{d}s^2_{\mathcal{SK}_c} = \partial\bar\partial\mathcal{K} \,, \qquad \mathcal{K} = -\log[\mathrm{i}(\bar X^\Lambda F_\Lambda - X^\Lambda \bar F_\Lambda)]$$

- Harmonic C-fields on X may be parametrized by the real periods

$$\zeta^\Lambda = \int_{\mathcal{A}^\Lambda} C, \quad \tilde\zeta_\Lambda = \int_{\mathcal{B}_\Lambda} C \,.$$

The perturbative metric III

- Large gauge transformations require that $C \equiv (\zeta^\Lambda, \tilde\zeta_\Lambda)$ takes values in the intermediate Jacobian torus

$$C \in T = H^3(X,\mathbb{R})/H^3(X,\mathbb{Z})$$

i.e. that $(\zeta^\Lambda, \tilde\zeta_\Lambda)$ have unit periodicities.

- T carries a canonical symplectic form and complex structure induced by the Hodge $\star_X$, hence a Kähler metric

$$\mathrm{d}s^2_T = -\frac{1}{2}(\mathrm{d}\tilde\zeta_\Lambda - \bar{\mathcal{N}}_{\Lambda\Lambda'}\mathrm{d}\zeta^{\Lambda'})\mathrm{Im}\mathcal{N}^{\Lambda\Sigma}(\mathrm{d}\tilde\zeta_\Lambda - \mathcal{N}_{\Sigma\Sigma'}\mathrm{d}\zeta^{\Sigma'})$$

where $\mathcal{N}$ (resp. τ) is the Weil (resp. Griffiths) period matrix,

$$\mathcal{N}_{\Lambda\Lambda'} = \bar\tau_{\Lambda\Lambda'} + 2\mathrm{i}\frac{[\mathrm{Im}\tau\cdot X]_\Lambda[\mathrm{Im}\tau\cdot X]_{\Lambda'}}{X^\Sigma\,\mathrm{Im}\tau_{\Sigma\Sigma'}X^{\Sigma'}}\,, \qquad \tau_{\Lambda\Sigma} = \partial_{X^\Lambda}\partial_{X^\Sigma}F$$

The tree-level metric

- At tree level, i.e. in the strict weak coupling limit $R = \infty$, the quaternion-Kähler metric on $\mathcal{M}$ is given by the *c*-map metric

$$ds^2_{\mathcal{M}} = \frac{4}{R^2}\,\mathrm{d}R^2 + 4\,\mathrm{d}s^2_{\mathcal{SK}} + \frac{\mathrm{d}s^2_T}{R^2} + \frac{1}{16R^4}D\sigma^2 \,.$$

where

$$D\sigma \equiv \mathrm{d}\sigma + \langle C, \mathrm{d}C\rangle = \mathrm{d}\sigma + \tilde\zeta_\Lambda \mathrm{d}\zeta^\Lambda - \zeta^\Lambda \mathrm{d}\tilde\zeta_\Lambda$$

Cecotti Girardello Ferrara; Ferrara Sabharwal

- The *c*-map (aka semi-flat) metric admits continuous isometries

$$T_{H,\kappa} : (C,\sigma) \mapsto (C+H, \sigma + 2\kappa + \langle C, H\rangle)$$

where $H \in H^3(X,\mathbb{R})$ and $\kappa \in \mathbb{R}$, satisfying the Heisenberg algebra

$$T_{H_1,\kappa_1}\,T_{H_2,\kappa_2} = T_{H_1+H_2,\kappa_1+\kappa_2+\frac{1}{2}\langle H_1,H_2\rangle}$$

The one-loop corrected metric I

- The one-loop correction deforms the metric on $\mathcal{M}$ into

$$ds^2_{\mathcal{M}} = 4\frac{R^2+2c}{R^2(R^2+c)}\,\mathrm{d}R^2 + \frac{4(R^2+c)}{R^2}\,\mathrm{d}s^2_{\mathcal{SK}} + \frac{\mathrm{d}s^2_T}{R^2}$$

$$+ \frac{2\,c}{R^4}\,e^{\mathcal{K}}\,|X^\Lambda \mathrm{d}\tilde\zeta_\Lambda - F_\Lambda \mathrm{d}\zeta^\Lambda|^2 + \frac{R^2+c}{16R^4(R^2+2c)}D\sigma^2\,.$$

  where $D\sigma = \mathrm{d}\sigma + \langle C, \mathrm{d}C\rangle + 8c\mathcal{A}$,

$$\mathcal{A} = \tfrac{\mathrm{i}}{2}(\mathcal{K}_a \mathrm{d}z^a - \mathcal{K}_{\bar a}\mathrm{d}\bar z^{\bar a})\,, \qquad c = -\frac{\chi(X)}{192\pi}$$

Antoniadis Minasian Theisen Vanhove; Gunther Hermann Louis;

Robles-Llana Saueressig Vandoren

- The one-loop correction to g_{RR} was computed by reducing the $t_8 t_8 R^4$ coupling in 10D on $\mathcal{X}$. The correction to $D\sigma$ can be obtained with less effort by reducing $B\wedge R^4$ couplings in 10D.

Topology I

- Large gauge transformations of the B and C fields act as

$$T_{H,\kappa}: \quad \begin{cases} \zeta^\Lambda & \mapsto \quad \zeta^\Lambda + n^\Lambda \\ \tilde\zeta_\Lambda & \mapsto \quad \tilde\zeta_\Lambda + m_\Lambda \\ \sigma & \mapsto \quad \sigma + \kappa - m_\Lambda \zeta^\Lambda + n^\Lambda \tilde\zeta_\Lambda \\ & \qquad -n^\Lambda m_\Lambda + m_\Lambda \theta^\Lambda - n^\Lambda \phi_\Lambda \end{cases}$$

  where $H \equiv (n^\Lambda, m_\Lambda) \in H^3(X, \mathbb{Z})$, $\Theta = (\theta^\Lambda, \phi_\Lambda) \in \mathcal{T}$ is a choice of characteristics and $\kappa \in \mathbb{Z}$.

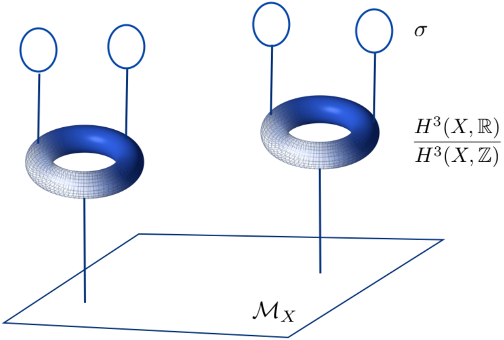
- The shift of σ is needed for the closure of the group action,

$$T_{H_1,\kappa_1} T_{H_2,\kappa_2} = T_{H_1+H_2,\kappa_1+\kappa_2+\frac{1}{2}\langle H_1,H_2\rangle + \frac{1}{2}\langle H_1,H_2\rangle}$$

Alexandrov Persson BP

Topology II

- Topologically, $\mathcal{M}$ is a $\mathbb{C}^\times$ bundle over the intermediate Jacobian,



D-instanton corrections I

- To all orders in $1/R$, the metric is flat along the twisted torus fibers. There are presumably no perturbative corrections beyond one-loop.

Robles Llana Saueressig Vandoren; Gunther Louis

- D-instanton corrections will break continuous isometries along the torus T. NS5-brane instantons will further break isometry along σ.
- Instantons are necessary to restore S-duality invariance, and to cure the curvature singularity at finite distance $R^2 = -2c$ when $\chi(X) > 0$.

D-instanton corrections II

- Euclidean D2-branes wrapping special Lagrangian submanifolds in integer homology class $\gamma = q_\Lambda \mathcal{A}^\Lambda - p^\Lambda \mathcal{B}_\Lambda \in H_3(X, \mathbb{Z})$ are expected to induce corrections of the form

$$\delta \mathrm{d}s^2|_{D2} \sim \bar{\Omega}(\gamma; z^a) \exp\left(-8\pi \frac{|Z_\gamma|}{g_{(4)}} - 2\pi\mathrm{i}\langle \gamma, C\rangle \right) + \dots$$

  Here $Z_\gamma \equiv e^{\mathcal{K}/2}(q_\Lambda X^\Lambda - p^\Lambda F_\Lambda)$ is the central charge, $\bar{\Omega}(\gamma; z^a)$ are the generalized DT invariants, roughly the number of SLAGs. The dots stand for multi-instantons.

- The exact form is essentially dictated by consistency with wall crossing, and best expressed in twistor space

Gaiotto Moore Neitze; Kontsevich Soibelman; Alexandrov BP Saueressig Vandoren

Outline

A lightning review of twistors I

- QK manifolds $\mathcal{M}$ are conveniently described via their twistor space $\mathbb{P}^1 \to \mathcal{Z} \to \mathcal{M}$, a complex contact manifold with real involution. Choosing a stereographic coordinate t on $\mathbb{P}^1$, the contact structure is the kernel of the local (1,0)-form

$$Dt = \mathrm{d}t + p_+ - \mathrm{i}p_3 t + p_- t^2$$

  where $p_3, p_\pm$ are the $SU(2)$ components of the Levi-Civita connection on $\mathcal{M}$. Dt is well-defined modulo rescalings.

- $\mathcal{Z}$ is further equipped with a Kähler-Einstein metric

$$\mathrm{d}s^2_{\mathcal{Z}} = \frac{|Dt|^2}{(1+t\bar{t})^2} + \frac{\nu}{4}\mathrm{d}s^2_{\mathcal{M}} \,, \qquad \nu = \frac{R(\mathcal{M})}{4d(d+2)}$$

  If $\mathcal{M}$ has negative scalar curvature, $\mathcal{Z}$ is pseudo-Kähler with signature $(2, \dim \mathcal{M})$.

Lebrun, Salamon

A lightning review of twistors II

- Rk: complex contact manifolds are projectivizations of complex symplectic cones. The $\mathbb{C}^\times$ bundle over $\mathcal{Z}$ is the hyperkähler cone associated to $\mathcal{M}$. The two approaches are equivalent.

Swann; de Wit Rocek Vandoren

- Locally, there always exist Darboux coordinates $(\Xi, \tilde\alpha) = (\xi^\Lambda, \tilde\xi_\Lambda, \tilde\alpha)$ and a “contact potential" Φ such that

$$2\, e^{\Phi} \frac{Dt}{\mathrm{i}t} = \mathrm{d}\tilde\alpha + \langle \Xi, \mathrm{d}\Xi \rangle = \mathrm{d}\tilde\alpha + \tilde\xi_\Lambda \mathrm{d}\xi^\Lambda - \xi^\Lambda \mathrm{d}\tilde\xi_\Lambda \ .$$

- The contact potential is independent of $\bar t$, and provides a Kähler potential for the Kähler metric on $\mathcal{Z}$ via $e^{K_{\mathcal{Z}}} = (1+t\bar t) e^{Re(\Phi)}/|t|$.

Alexandrov BP Saueressig Vandoren

A lightning review of twistors III

- The global contact geometry on $\mathcal{Z}$ can be specified by a set of complex contact transformations on overlaps of Darboux coordinate patches.
- By the moment map construction, continuous isometries of $\mathcal{M}$ are in 1-1 correspondence with classes in $H^0(\mathcal{Z}, \mathcal{O}(2))$. In particular, any continuous isometry of $\mathcal{M}$ can be lifted to a holomorphic action on $\mathcal{Z}$.

Salamon; Galicki Salamon

- Infinitesimal deformations of $\mathcal{M}$ lift to deformations of the complex contact transformations between Darboux coordinate patches on $\mathcal{Z}$, hence are classified by $H^1(\mathcal{Z}, \mathcal{O}(2))$.

Lebrun; Alexandrov BP Saueressig Vandoren

Twistor description of the perturbative metric

- For the one-loop corrected HM metric, the following Darboux coordinates do the job, away from the north and south poles $t = 0, \infty$:

$$\begin{aligned} \Xi_{\rm sf} &= C + 2\sqrt{R^2 + c}\, e^{\mathcal{K}/2} \left[t^{-1}\Omega - t\,\bar\Omega \right] , \quad \Phi_{\rm sf} = 2\log R\,, \\ \tilde\alpha_{\rm sf} &= \sigma + 2\sqrt{R^2 + c}\, e^{\mathcal{K}/2} \left[t^{-1}\langle \Omega, C\rangle - t\,\langle \bar\Omega, C\rangle \right] - 8\mathrm{i}c\log t \end{aligned}$$

Neitzke BP Vandoren; Alexandrov

- The isometry $T_{H,\kappa}$ acts holomorphically on $\mathcal{Z}$ by

$$(\Xi, \tilde\alpha) \mapsto (\Xi + H, \tilde\alpha + 2\kappa + \langle \Xi, H\rangle)$$

- Modding out by large gauge transformations $T_{H,\kappa}$, $\mathcal{Z}$ becomes a complexified twisted torus $\mathbb{C}^\times \ltimes [H^3(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^\times]$.

D-instantons in twistor space I

- D-instanton corrections to $\mathcal{Z}$ are essentially dictated by wall crossing. Recall that the Kontsevich-Soibelman wall-crossing formula requires that the product

$$\prod_\gamma U_\gamma \,, \qquad U_\gamma \equiv \exp\left(\Omega(\gamma; t^a) \sum_{d=1}^{\infty} \frac{e_{d\gamma}}{d^2} \right) \,,$$

ordered such that $\arg(Z_\gamma)$ decreases from left to right, stays invariant across the wall. Here $\Omega(\gamma; t^a)$ are generalized DT invariants, and e_γ satisfy the Lie algebra

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle\gamma_1,\gamma_2\rangle} \langle\gamma_1, \gamma_2\rangle \, e_{\gamma_1+\gamma_2} \,.$$

D-instantons in twistor space II

- Using the quadratic refinement $\lambda(\gamma) = e^{\mathrm{i}\pi p^\Lambda q_\Lambda + 2\pi\mathrm{i}(p^\Lambda \phi_\Lambda - q_\Lambda \theta^\Lambda)}$ to cancel the phase, one can represent e_γ as a contact-Hamiltonian vector field on $\mathcal{Z}$,

$$\lambda(\gamma)\, e_\gamma = (\partial_{\xi^\Lambda} \mathcal{X}_\gamma)\, \partial_{\tilde\xi_\Lambda} - (\partial_{\tilde\xi_\Lambda} \mathcal{X}_\gamma)\, \partial_{\xi^\Lambda} + 2\mathrm{i}[(2 - \xi^\Lambda \partial_{\xi^\Lambda} - \tilde\xi_\Lambda \partial_{\tilde\xi_\Lambda})\mathcal{X}_\gamma]\, \partial_{\tilde\alpha}$$

  where

$$\mathcal{X}_\gamma = \mathrm{E}^{\langle \Xi, \gamma \rangle} = \mathrm{E}^{q_\Lambda \xi^\Lambda - p^\Lambda \tilde\xi_\Lambda}$$

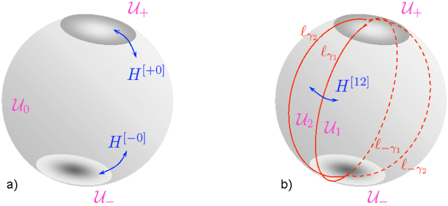
- Exponentiating, U_γ implements the contact transformation

$$\mathcal{X}_{\gamma'} \mapsto \mathcal{X}_{\gamma'} (1 - \lambda(\gamma)\mathcal{X}_\gamma)^{\langle \gamma, \gamma' \rangle\, \Omega(\gamma)}\,, \qquad \tilde\alpha \mapsto \tilde\alpha - \frac{1}{2\pi^2} \Omega(\gamma)\, L[\lambda(\gamma)\mathcal{X}_\gamma]$$

  where $L(z) = \sum_{n=1}^\infty \frac{z^n}{n^2} + \frac{1}{2} \log z\, \log(1-z)$ is Rogers' dilogarithm.

Alexandrov Saueressig BP Vandoren; Alexandrov Persson BP

- By analogy with GMN, the D-instanton corrected twistor space is obtained by gluing Darboux coordinate patches along BPS rays $\ell_\pm = \{t \; : \; Z(\gamma; z^a)/t \in \pm i\mathbb{R}^+\}$, using the contact transf. U_γ:



- The consistency of this prescription across lines of marginal stability is guaranteed by the KS formula. The resulting metric, including multi-instanton corrections, is smooth across the walls.

Gaiotto Moore Neitzke

Multi-instanton corrections

- The gluing conditions for $\Xi = (\xi^\Lambda, \tilde{\xi}_\Lambda)$ can be summarized by integral equations

$$\Xi = \Xi_{\rm sf} - \frac{1}{8\pi^2} \sum_\gamma \Omega(\gamma) \langle \cdot, \gamma \rangle \int_{\ell_\gamma} \frac{dt'}{t'} \frac{t+t'}{t-t'} \mathrm{Li}_1 \left[\lambda(\gamma) \, \mathrm{E}^{-\langle \Xi(t'), \gamma \rangle} \right],$$

where $\mathrm{Li}_1(x) \equiv -\log(1-x)$. Similar eqs allowing to compute $\tilde{\alpha}$, Φ once Ξ is known.

- These eqs are formally identical to Zamolodchikov's Y-system in studies of integrable models.

Gaiotto Moore Neitzke; Alexandrov Roche

- These eqs can be solved iteratively, by first substituting $\Xi \to \Xi_{\rm sf}$ on the rhs, integrating, etc. leading to an infinite (divergent) series of multi-instanton corrections.

Outline

HM moduli space in type IIB I

- The HM moduli space in type IIB compactified on a CY 3-fold Y is a QK manifold $\mathcal{M}$ of real dimension $4(h_{1,1}(Y)+1)$ describing
  1. the 4D dilaton $R \equiv 1/g_4$,
  2. the complexified Kähler moduli $z^a = b^a + \mathrm{i}t^a \in \mathcal{SK}$
  3. the RR scalars $C \in T = H^{\mathrm{even}}(Y,\mathbb{R})/H^{\mathrm{even}}(Y,\mathbb{Z})$
  4. the NS axion σ dual to B-field in 4 dimensions
- At tree level, i.e. in the strict weak coupling limit $R=\infty$, the quaternion-Kähler metric on $\mathcal{M}$ is given by the *c*-map metric

$$ds^2_{\mathcal{M}} = \frac{4}{R^2}\,\mathrm{d}R^2 + 4\,\mathrm{d}s^2_{\mathcal{SK}} + \frac{\mathrm{d}s^2_T}{R^2} + \frac{1}{16R^4}D\sigma^2\,.$$

where

$$D\sigma \equiv \mathrm{d}\sigma + \langle C, \mathrm{d}C\rangle = \mathrm{d}\sigma + \tilde\zeta_\Lambda \mathrm{d}\zeta^\Lambda - \zeta^\Lambda \mathrm{d}\tilde\zeta_\Lambda$$

Cecotti Girardello Ferrara; Ferrara Sabharwal

HM moduli space in type IIB II

- At large volume, the metric on $\mathcal{SK}$ is governed by the prepotential

$$F(X) = -\frac{1}{6}\kappa_{abc}\frac{X^a X^b X^c}{X^0} + \chi(Y)\,\frac{\zeta(3)(X^0)^2}{2(2\pi\mathrm{i})^3} + F_{\mathrm{GW}}(X)$$

  where κ_{abc} is the cubic intersection form and F_{GW} are Gromov-Witten instanton corrections:

$$F_{\mathrm{GW}}(X) = -\frac{(X^0)^2}{(2\pi\mathrm{i})^3}\sum_{k_a\gamma^a\in H_2^+(Y)} n^{(0)}_{k_a}\,\mathrm{Li}_3\left[\mathrm{E}^{k_a\frac{X^a}{X^0}}\right],$$

- The same one-loop correction appears as in type IIA, with opposite sign.

Classical S-duality I

- In the large volume, classical limit, the metric is invariant under

$$\tau \mapsto \frac{a\tau+b}{c\tau+d}\,, \qquad t^a \mapsto t^a|c\tau+d|\,, \qquad \tilde{c}_a \mapsto \tilde{c}_a$$
$$\begin{pmatrix} c^a \\ b^a \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} c^a \\ b^a \end{pmatrix}\,, \qquad \begin{pmatrix} \tilde{c}_0 \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \tilde{c}_0 \\ \psi \end{pmatrix}\,,$$

where $\tau = \tau_1 + \mathrm{i}\tau_2$ is the 10D axio-dilaton and

$$\zeta^0 = \tau_1\,, \quad \zeta^a = -(c^a - \tau_1 b^a)\,,$$
$$\tilde{\zeta}_a = \tilde{c}_a + \frac{1}{2}\kappa_{abc}\, b^b(c^c - \tau_1 b^c)\,, \quad \tilde{\zeta}_0 = \tilde{c}_0 - \frac{1}{6}\kappa_{abc}\, b^a b^b (c^c - \tau_1 b^c)\,,$$
$$\sigma = -2\psi - \tau_1 \tilde{c}_0 + \tilde{c}_a(c^a - \tau_1 b^a) - \frac{1}{6}\kappa_{abc}\, b^a c^b (c^c - \tau_1 b^c)\,.$$

- Translations along $(b^a, c^a), \tilde{c}_a, (c_0, \psi)$ form a 3-step nilpotent algebra N.

S-duality in twistor space I

- The action of S-duality on $\mathcal{M}$, combined with a suitable $U(1)$ rotation along the fiber,

$$z \mapsto \frac{c\bar{\tau}+d}{|c\tau+d|}\, z, \qquad z \equiv \frac{t+\mathrm{i}}{t-\mathrm{i}},$$

lifts to a holomorphic action on $\mathcal{Z}$ via (here $\alpha = -\frac{1}{2}(\tilde{\alpha} + \xi^\Lambda \tilde{\xi}_\Lambda)$)

$$\xi^0 \mapsto \frac{a\xi^0+b}{c\xi^0+d}\,, \quad \xi^a \mapsto \frac{\xi^a}{c\xi^0+d}\,, \quad \tilde{\xi}_a \mapsto \tilde{\xi}_a + \frac{c}{2(c\xi^0+d)}\kappa_{abc}\xi^b\xi^c\,,$$

$$\begin{pmatrix} \tilde{\xi}_0 \\ \alpha \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \tilde{\xi}_0 \\ \alpha \end{pmatrix} + \frac{1}{6}\,\kappa_{abc}\xi^a\xi^b\xi^c \begin{pmatrix} c^2/(c\xi^0+d) \\ -[c^2(a\xi^0+b)+2c]/(c\xi^0+d)^2 \end{pmatrix}$$

- (ξ^0, ξ^a) transform like (modular parameter, elliptic variable). $\mathrm{E}^{p^a \tilde{\xi}_a}$ transforms like the automorphic factor of a Jacobi form.
- Note that S-duality fixes the points $t = \pm\mathrm{i}$ along the $\mathbb{P}^1$ fiber.

S-duality and D1-F1-D(-1) instantons I

- Continuous S-duality is broken by Gromov-Witten instantons at tree level and by the one-loop correction.
- In the large volume limit, retaining D1-F1-D(-1) instantons, S-duality mixes (one-loop,D(-1)) and (D1,F1). It was shown that $SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$ remains unbroken provided

$$\Omega(0,0,0,q_0) = -\chi(Y) \,, \qquad \Omega(0,0,q_a,q_0) = n^{(0)}_{q_a}$$

Robles-Llana Roček Saueressig Theis Vandoren

- The (GMN-type) 'type IIA' Darboux coordinates are not covariant, but they can be mapped by a contact transformation (and Poisson resummation on q_0) to a set of 'type IIB' Darboux coordinates which transform as above.

Alexandrov Saueressig

S-duality and D1-F1-D(-1) instantons II

- In the IIB frame, the twistor space is covered by open sets $U_{m,n}$ centered around $m\xi^0 + n = 0$, with transition functions $U_{0,0} \mapsto U_{m,n}$ generated by

$$G_{m,n}(\xi^0, \xi^a) = -\frac{\mathrm{i}}{(2\pi)^3} \sum_{q_a \geq 0} n^{(0)}_{q_a} \begin{cases} \frac{e^{-2\pi \mathrm{i} m q_a \xi^a}}{m^2(m\xi^0+n)}\,, & m \neq 0 \\ (\xi^0)^2\, \frac{e^{2\pi \mathrm{i} n q_a \xi^a/\xi^0}}{n^3}\,, & m = 0 \end{cases}$$

- Under $SL(2,\mathbb{Z})$, $G_{m,n}$ are mapped into each other,

$$\begin{pmatrix} m' \\ n' \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}\,, \quad G_{m,n} \mapsto \frac{G_{m',n'}}{c\xi^0 + d} + \text{reg.}$$

UP to a term regular in $U_{m',n'}$.

- The contact potential can be written in terms of Kronecker-Eisenstein series, or elliptic dilogarithm.

S-duality and D3-D1-F1-D(-1) instantons I

- Since the same BPS invariants govern D3-D1-D(-1) instantons in IIB/$\mathcal{X}$ and D4-D2-D0 black holes in IIA/$\mathcal{X}$, S-duality is expected to follow from the modularity of the D4-D2-D0 black hole partition

$$\mathcal{Z}_{\mathrm{BH}}(\tau, y^a) = \sum_{q_a, q_0} \Omega^{\mathrm{MSW}}(p^a, q_a, q_0)\, \mathrm{E}^{-(q_0+\frac{1}{2}q_-^2)\tau - q_+^2\bar\tau + q_a y^a}$$

$$= \mathrm{Tr}'(2J_3)^2(-1)^{2J_3}\, \mathrm{E}^{\left(L_0-\frac{c_L}{24}\right)\tau - \left(\bar L_0 - \frac{c_R}{24}\right)\bar\tau + q_a y^a}$$

where q_+, q_- are the projections of q_a on $H^{1,1}$ and $(H^{1,1})^\perp$.

- When p^a is a very ample primitive divisor, $\mathcal{Z}_{\mathrm{BH}}$ is the modified elliptic genus of the MSW superconformal CFT, a multivariate Jacobi form of weight $(-\frac{3}{2}, \frac{1}{2})$, index $\kappa_{ab} = \kappa_{abc} p^c$ and multiplier system $v_\eta^{c_{2a} p^a}$.

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Denef Moore; Gaiotto Strominger Yin; de Boer Cheng Dijkgraaf Manschot Verlinde

S-duality and D3-D1-F1-D(-1) instantons II

- Spectral flow invariance of the SCFT implies that $\Omega^{\rm MSW}(p^a, q_a, q_0)$ depends only on p^a, $\hat{q}_0 \equiv q_0 - \frac{1}{2} q_a \kappa^{ab} q_b$ and on the residue μ^a of $q_a \in \Lambda^* + \frac{1}{2} p$ modulo Λ. Thus

$$\mathcal{Z}_{\rm BH}(\tau, y^a) = \sum_{\mu \in \Lambda^*/\Lambda} h_{p^a,\mu_a}(\tau)\, \overline{\theta_{p^a,\mu_a}(\tau, y^a, p^a)}\,,$$

where θ_{p^a,μ_a} is a signature $(1, b_2 - 1)$ Siegel-Narain theta series,

$$\theta_{p^a,\mu_a}(\tau, y^a, t^a) = \sum_{k \in \Lambda + \mu + \frac{1}{2} p} (-1)^{p \cdot k}\, \mathrm{E}^{\frac{1}{2}(k_+)^2 \tau + \frac{1}{2}(k_-)^2 \bar{\tau} + k \cdot y}$$

and

$$h_{p^a,\mu_a} = \sum_{\hat{q}_0} \Omega^{\rm MSW}(p^a, \mu_a, \hat{q}_0)\, \mathrm{E}^{-\hat{q}_0 \tau}$$

is a weight $(-\frac{b_2}{2} - 1, 0)$ vector-valued modular form.

S-duality and D3-D1-F1-D(-1) instantons III

- There is an important catch: the MSW degeneracies $\Omega^{\rm MSW}_{p^a,q_a,q_0}$ agree with the generalized DT invariants only at the 'large volume attractor point'

$$\Omega^{\rm MSW}(p^a, q_a, q_0) = \lim_{\lambda\to+\infty} \bar\Omega\,(0, p^a, q_a, q_0; b^a(\gamma) + {\rm i}\lambda\, t^a(\gamma))$$

- Away from this point, DT invariants get contributions from bound states of MSW micro-states. To exhibit modular invariance, we need to first express the generalized DT invariants in terms of MSW invariants, and then do the multi-instanton expansion in powers of $\Omega^{\rm MSW}(p^a, q_a, q_0)$.
- We shall restrict to the one-instanton approximation, effectively identifying $\bar\Omega(0, p^a, q_a, q_0; z^a) = \Omega^{\rm MSW}(p^a, q_a, q_0)$. Moreover we work in the large volume limit, zooming around $t = \pm i$ $(z = 0, \infty)$.

S-duality and D3-D1-F1-D(-1) instantons IV

- By expanding the integral equations to first order in $\Omega^{\rm MSW}$, and allowing first order corrections to the mirror map between $\zeta^\Lambda, \tilde\zeta_\Lambda, \sigma$ and $c^a, \tilde c_a, \tilde c_0, \sigma$, one finds

$$\delta e^\Phi = \frac{\tau_2\, e^{-2\pi S_{\rm cl}}}{16\pi^2\sqrt{2\tau_2\, p\cdot t^2}}\,\mathcal{D}_{-\frac{3}{2}} \sum_{\mu\in\Lambda^*/\Lambda} h_{\boldsymbol{p},\boldsymbol{\mu}}(\tau)\,\overline{\theta_{\boldsymbol{p},\boldsymbol{\mu}}(\tau, t^a, b^a, c^a)} + {\rm c.c.}$$

$$\delta\xi^0 = 0\,,\quad \delta\xi^a = 2\pi {\rm i} p^a\,\mathcal{J}_p(z)\,,\quad \delta\tilde\xi_a = -D_a\,\mathcal{J}_p(z)\,,\quad \delta\tilde\xi_0 = \dots$$

where $S_{\rm cl} = \frac{\tau_2}{2}\,\kappa_{abc}p^at^bt^c - {\rm i}\,\tilde c_a p^a$ is the classical D3-brane action,

$$\mathcal{J}_p(z) = \sum_{q_\Lambda}\int_{\ell_\gamma}\frac{{\rm d}z'}{(2\pi)^3{\rm i}(z'-z)}\,\Omega^{\rm MSW}(p^a, q_a, q_0)\,{\rm E}^{p^a\tilde\xi_a - q_\Lambda\xi^\Lambda}\,,$$

- The correction to the contact potential is a modular derivative of the MSW elliptic genus, with the correct modular weight $(-\frac{1}{2}, -\frac{1}{2})$.

S-duality and D3-D1-F1-D(-1) instantons V

- Unlike δe^{Φ}, corrections to the Darboux coordinates have a modular anomaly, best exposed by rewriting the Penrose-type integral along z as an Eichler integral

$$\mathcal{J}_{\boldsymbol{p}}(z) = \frac{\mathrm{i}\, e^{-2\pi S_{\mathrm{cl}}}}{8\pi^2} \sum_{\boldsymbol{\mu}\in\Lambda^*/\Lambda} h_{\boldsymbol{p},\boldsymbol{\mu}}(\tau) \int_{\bar\tau}^{-\mathrm{i}\infty} \frac{\overline{\Upsilon_{\boldsymbol{\mu}}(w,\bar\tau;\bar z)}\,\mathrm{d}\bar w}{\sqrt{\mathrm{i}(\bar w-\tau)}}$$

where, restricting to $z=0$ for simplicity,

$$\overline{\Upsilon_{\boldsymbol{\mu}}(w,\bar\tau;0)} = \sum_{\boldsymbol{k}\in\Lambda+\boldsymbol{\mu}+\frac{1}{2}\boldsymbol{p}} (-1)^{\boldsymbol{k}\cdot\boldsymbol{p}}\,(k+b)_+ \times \mathrm{E}^{-\frac{1}{2}(\boldsymbol{k}+\boldsymbol{b})_+^2\,\bar w - \frac{1}{2}(\boldsymbol{k}+\boldsymbol{b})_-^2\,\tau + \boldsymbol{c}\cdot(\boldsymbol{k}+\frac{1}{2}\boldsymbol{b})}.$$

S-duality and D3-D1-F1-D(-1) instantons VI

- Eichler integrals of an analytic modular form $F(\tau,\bar\tau)$ of weight $(\mathfrak{h},\bar{\mathfrak{h}})$ (known as the shadow) are defined by

$$\Phi(\tau) = \int_{\bar\tau}^{-\mathrm{i}\infty} \frac{F(\tau,\bar w)\,\mathrm{d}\bar w}{[\mathrm{i}(\bar w-\tau)]^{2-\bar{\mathfrak{h}}}}$$

They transform with modular weight $(\mathfrak{h}+2-\bar{\mathfrak{h}},0)$, up to modular anomaly given by a period integral,

$$\Phi(\gamma\tau) = (c\tau+d)^{\bar{\mathfrak{h}}+2-\mathfrak{h}}\left(\Phi(\tau) - \int_{-d/c}^{-\mathrm{i}\infty} \frac{F(\tau,\bar w)\,\mathrm{d}\bar w}{[\mathrm{i}(\bar w-\tau)]^{2-\bar{\mathfrak{h}}}}\right).$$

- In particular, $\mathscr{J}_{\boldsymbol{p}}(z)$ transforms with modular weight $(-1,0)$, up to modular anomaly of the form above.

S-duality and D3-D1-F1-D(-1) instantons VII

- Miraculously, the modular anomalies in the Darboux coordinates $\xi^a, \tilde{\xi}_a, \tilde{\xi}_0, \alpha$ can be absorbed all at once by a contact transformation generated by

$$H = \frac{1}{8\pi^2} \mathrm{E}^{p^a \tilde{\xi}_a} \sum_{\mu \in \Lambda^*/\Lambda} h_{p^a,\mu_a}(\xi^0)\, \Theta_{p^a,\mu_a}(\xi^0, \xi^a)$$

where Θ_{p^a,μ_a} is Zwegers' indefinite theta series, viewed as a holomorphic function in twistor space,

$$\theta_{p^a,\mu_a}(\xi^0, \xi^a) = \sum_{k \in \Lambda + \mu + p/2} \left(\mathrm{sign}[(k+b)\cdot t] - \mathrm{sign}[(k+b)\cdot t_1]\right) \times (-1)^{p\cdot k}\, \mathrm{E}^{-k_a \xi^a - \frac{1}{2}\xi^0\, k_a \kappa^{ab} k_b}$$

Here t_1 is an arbitrary point on the boundary of the Kahler cone.

S-duality and D3-D1-F1-D(-1) instantons VIII

- The fact that h_{p^a,μ_a} transforms with multiplier system $v_\eta^{p^a c_{2,a}}$ implies that $\tilde{c}_a$ must transform with an additional shift $\tilde{c}_a \mapsto \tilde{c}_a - c_{2,a}\varepsilon(g)$ under S-duality, where

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)/\eta(\tau) = e^{2\pi\mathrm{i}\epsilon(g)}(c\tau+d)^{1/2}\,.$$

- Amusingly, the holomorphic theta series provides the modular completion of the Eichler integral, rather than the other way around ! The latter arises as a Penrose-type integral along the fiber.
- It would be desirable to understand the physical significance of the reference point t_1.

Outline

Five-brane instantons I

- NS5-brane instantons with charge $k \in \mathbb{Z}$ are expected to produce corrections to the metric of the form

$$\delta \mathrm{d}s^2|_{\mathrm{NS5}} \sim \exp\left(-4\pi \frac{|k|}{g_{(4)}^2} - \mathrm{i}k\pi\sigma\right) \mathcal{Z}^{(k)}(z^a, C) ,$$

  where $\mathcal{Z}^{(k)} = \mathrm{Tr}[(2J_3)^2(-1)^{2J_3}]$ is the (twisted) partition function of the world-volume theory on a stack of k five-branes.
- Recall that the type IIA NS5-brane supports a self-dual 3-form flux, together with its SUSY partners. The partition function of a self-dual form is known to be a holomorphic section of a non-trivial line bundle $\mathcal{L}^{k}_{\mathrm{NS5}}$ over the space of metrics and C fields. This is consistent with the topology of the NS axion circle bundle.

Witten; Henningson Nilsson Salomonson; Belov Moore; . . .

Five-brane instantons II

- This means that $\mathcal{Z}^{(k)}(z^a, C)$ satisfies the twisted periodicity condition

$$\mathcal{Z}^{(k)}(z^a, C+H) = \lambda^k(H)\, e^{\mathrm{i}\pi k \langle H, C\rangle}\, \mathcal{Z}^{(k)}(z^a, C)$$

where $\lambda(H) : H_3(\mathcal{X}, \mathbb{Z}) \to U(1)$ is a quadratic refinement of the symplectic pairing, (here $H = (m_\Lambda, n^\Lambda)$)

$$\lambda(H+H') = (-1)^{\langle H,H'\rangle}\, \lambda(H)\, \lambda(H')\ , \quad \lambda(H) = e^{-\mathrm{i}\pi m^\Lambda n_\Lambda + 2\pi\mathrm{i}\langle H, \Theta\rangle}$$

where $\Theta = (\theta, \phi) \in \mathcal{T}$ are a choice of characteristics.

- Holomorphic sections of $(\mathcal{L}_\Theta)^k$ are Siegel theta series of rank $b_3(\mathcal{X})$, level $k/2$, but holomorphy holds only in large volume limit.

S-duality and D5-NS5 instantons

- S-duality relates D5 and NS5. Starting from the known form of D5-D3-D1-D(-1) corrections, one may construct a Poincaré-type series to obtain the contributions from k NS5-branes at linear order.
- The generating function of NS5-instantons is a non-Gaussian, non-Abelian generalization of the five-brane partition function

$$H^{(k)}_{\rm NS5}(\xi,\tilde\xi,\tilde\alpha) = \frac{1}{4\pi^2} \sum_{\substack{\mu\in(\Gamma_m/|k|)/\Gamma_m \\ n\in\Gamma_m+\mu+\theta}} H^{(k,\mu)}_{\rm NS5}\left(\xi^\Lambda - n^\Lambda\right) \, \mathrm{E}^{kn^\Lambda(\tilde\xi_\Lambda-\phi_\Lambda)-\frac{k}{2}(\tilde\alpha+\xi^\Lambda\tilde\xi_\Lambda)}.$$

- For $k=1$, one recovers the topological string amplitude:

$$H^{(1)}_{\rm NS5}(\xi^\Lambda) = \left(\xi^0\right)^{-1-\frac{\chi(Y)}{24}} \, [M(e^{2\pi i/\xi^0})]^{-\chi(Y)/2} \, \Psi^{\rm top}_{\mathbb{R}}(\xi^\Lambda)\,.$$

Alexandrov Persson Pioline

Outline

Conclusion I

- The hypermultiplet moduli space of $\mathcal{N}=2$ string vacua is a fascinating subject, which combines many trades in mathematics (algebraic geometry, symplectic geometry, number theory, etc).
- Combining twistor techniques with S-duality and mirror symmetry has lead to a complete and beautiful picture for D-instanton corrections. NS5-brane instantons are still mysterious beyond linear order, yet they are in principle determined by S-duality...
- Physically, the metric on $\mathcal{M}$ provides a grand-canonical partition function for $\mathcal{N}=2$ black holes, supplemented with NS5/KKM corrections. The latter should fix the ambiguity of the divergent series, $\sum_Q \Omega(Q) e^{-RQ}$.

BP Vandoren

Conclusion II

- So far, we have parametrized the metric in terms of the BPS invariants $\Omega(\gamma)$. It would be very interesting if one could compute those, e.g. by postulating additional automorphic properties (e.g. $SL(3,\mathbb{Z})$ or $SU(2,1)$)

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- In heterotic string compactified on $K3 \times T^2$, the HM moduli space parametrizes the space of metrics and bundles on $K3$, and should be entirely determined at string tree-level. Unfortunately, little is known about it !

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