Progress on hypermultiplet moduli spaces

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Introduction I

- Understanding the vector multiplet moduli space $\mathcal{M}_V$ of gauge theories and string vacua with $\mathcal{N} = 2$ SUSY in 4 dimensions (8 supercharges) has given key insights into non-perturbative physics:
  1. Exact resummations of gauge instantons: *Seiberg Witten, ...*
  2. Classical mirror symmetry: *Candelas de la Ossa Green Parks, ...*
  3. String dualities: *Kachru Vafa, ...*
- The special Kähler metric on $\mathcal{M}_V$is governed by a holomorphic function, the prepotential, determined by its behavior under monodromies around conifold-type singularities.

Introduction II

- The hypermultiplet moduli space $\mathcal{M}_H$ has been comparatively less studied, yet it also carries crucial physical and mathematical information, e.g.
    1. the HK metric on the Coulomb branch of $\mathcal{N} = 2$ gauge theories on $\mathbb{R}^3 \times S^1$ encodes the spectrum of BPS monopoles in $\mathbb{R}^4$;
    2. the QK metric on the HM moduli space of type II string theory compactified on a Calabi-Yau 3-fold X receives D-instanton and NS5-brane corrections, determined by geometric invariants of X;
- Progress has being hampered by the absence of a convenient parametrization of HK and QK metrics. Using twistor methods, they can still be described by holomorphic data, though the relation to the actual metric is rather less direct.

Introduction III

- Twistor methods have been around in the physics literature under the name projective superspace or harmonic superspace, but their power has only started to be more widely appreciated recently.

Hitchin Karlhede Lindström Rocek; Galperin Ivanov Ogievetsky Sokatchev

- In particular, it has become clear that HK/QK geometry is the correct framework for understanding wall-crossing formulae for governing the BPS spectrum in $N = 2$ gauge theories and supergravity.

Denef Moore,Kontsevich Soibelman; Gaiotto Neitzke Moore

Introduction IV

- The HM moduli space $\mathcal{M}_H$ in type II string theory compactified on a CY three-fold show should be the physical framework for a quantum version of mirror symmetry, which must weave together homological mirror symmetry, modularity and possibly new mathematics linked to NS5-branes.
- Today, I will review recent progress towards understanding $\mathcal{M}_H$, based in part on my own work with Alexandrov, Saueressig and Vandoren.

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Outline

1. Classical and homological mirror symmetry
2. The perturbative hypermultiplet moduli space
3. Twistor methods for quaternion-Kähler spaces
4. The non-perturbative hypermultiplet moduli space

Outline

Set-up I

- Consider type IIA string theory compactified on a Calabi-Yau three-fold X. The low energy physics is described by $\mathcal{N}=2, D=4$ (ungauged) supergravity, with $n_V = h^{1,1}(X)$ vector multiplets and $n_H = h^{2,1}(X)+1$ hypermultiplets.
- A vector multiplet (VM) consists of one complex-valued field t^a and one 1-form A^a_μ (hence its name), plus fermionic fields. A hypermultiplet (HM) consists of one quaternion-valued field q^Λ, plus fermions.
- The massless scalar fields $(t^a(x^\mu), q^\Lambda(x^\mu))$ provide a map from $D=4$ Minkowski space time into a Riemannian manifold $\mathcal{M}$, known as the moduli space. $\mathcal{M} = \mathcal{M}_V \times \mathcal{M}_H$ splits into the product of a projective special Kähler (PSK) manifold $\mathcal{M}_V$, of real dimension $2n_V$, and a quaternion-Kähler (QK) manifold $\mathcal{M}_H$, of real dimension $4n_H$.

Set-up II

- $\mathcal{M}_V \equiv \mathcal{SK}_K(X)$ parametrizes the complexified Kähler structure of X, while $\mathcal{M}_H \equiv \mathcal{QK}_C(X)$ parametrizes the complex structure of X, schematically

$$t^a = \int_{\gamma^a} B + \mathrm{i}\, J = b^a + \mathrm{i}\, j^a \,, \qquad q^\Lambda = \int_{\gamma^\Lambda} \Omega + j\,\mathcal{R}$$

  where (J, Ω) are the Kähler and (3,0) form, $(B, \mathcal{R})$ are the NS 2-form and RR multiform, γ^a a basis of $H_2(X, \mathbb{Z})$ and γ^Λ a basis of $H_3(X, \mathbb{Z})$. *Ω is normalized such that $\int_{\gamma^0} \Omega = \sigma + iV/g_s^2 l_s^6$, where V is the volume of X and g_s the string coupling.*

- The goal is to compute the Riemannian metric on $\mathcal{M}$ from data about X. String theory provides an asymptotic expansion in powers of the string coupling constant g_s, which a particular coordinate on $\mathcal{M}_H$. The main difficulty is in understanding non-perturbative effects of order e^{-1/g_s} or smaller.

$\mathcal{M}_V$ and classical mirror symmetry I

- The VM moduli space $\mathcal{M}_V$ is very well understood. By definition, its metric is independent of g_s, so can be computed in classical string theory. Still, it depends on the symplectic structure of X in a very non-trivial way.
- Since $\mathcal{M}_V$ is a projective special Kähler manifold, its geometry is encoded in the prepotential $F(X^\Lambda)$, a holomorphic function of projective coordinates X^Λ, $t^a = X^a/X^0$, homogeneous of degree two. Its third derivative $F_{\Lambda\Sigma\Xi}$ encodes the Yukawa couplings in the SUGRA action.

$\mathcal{M}_V$ and classical mirror symmetry II

- In the limit $V \gg l_s^6$, F is determined by the intersection product $C_{abc} = \int_X J_a J_b J_c$ in $H_4(X)$ and the Euler number χ. In addition, there are exponentially suppressed corrections (worldsheet instantons); here $\mathrm{e}^q = e^{2\pi i q_a X^a / X^0}$:

$$F = -C_{abc} \frac{X^a X^b X^c}{6X^0} + \chi\zeta(3) \frac{(X^0)^2}{2(2\pi i)^3} - \frac{(X^0)^2}{(2\pi i)^3} \sum_{q \in H_2^+(X)} N_{0,q}\, \mathrm{e}^q$$

- $N_{0,q}$ are rational numbers known as the genus 0 Gromov-Witten invariants. Defining $n_{0,q}$ via the multi-covering formula

$$\sum_q N_{0,q}\, \mathrm{e}^q = \sum_{q, d \geq 1} n_{0,q} \frac{\mathrm{e}^{dq}}{d^3}$$

The integers $n_{0,q}$ count the number of rational curves in homology class q. *They can be used to define the quantum cohomology ring of X.*

$\mathcal{M}_V$ and classical mirror symmetry III

- The Gromov-Witten invariants $N_{0,q}$ are most conveniently computed using (classical) mirror symmetry. Recall that for any (non-rigid) CY threefold X, there exists a mirror Calabi-Yau Y, such that $h_{1,1}(X) = h_{2,1}(Y)$, $h_{2,1}(X) = h_{1,1}(Y)$; if X is fibered by T^3, Y is fibered by T-dual/Mukai-transformed T^3).

Candelas et al; Strominger Yau Zaslow

- Mirror symmetry requires that $\mathcal{M}_V^{IIA}(X) = \mathcal{M}_V^{IIB}(Y)$, so $\mathcal{SK}_K(X) = \mathcal{SK}_C(Y)$. The prepotential $F(X^\Lambda)$ follows from period integrals of the (3,0) form Ω on Y:

$$X^\Lambda = \int_{\gamma^\Lambda} \Omega\,, \qquad F_\Lambda = \int_{\gamma_\Lambda} \Omega = \partial_\Lambda F\,,$$

where $\gamma^\Lambda, \gamma_\Lambda$ is a symplectic basis of $H_3(Y,\mathbb{Z})$, adapted to the point of maximal unipotent monodromy.

BPS spectrum and homological mirror symmetry I

- Mirror symmetry requires not only $\mathcal{M}_V^{IIA}(X) = \mathcal{M}_V^{IIB}(Y)$, but also that the full type IIA/X and type IIB/Y string theories be equivalent. In particular, the spectrum of BPS states should match.
- BPS states in type IIA/X are obtained by wrapping $D0, D2, D4, D6$ branes on complex submanifolds of X. More generally, they are realized as coherent sheaves on X; even more accurately, as elements in the derived category of coherent sheaves $DCoh(X)$.

Douglas

- BPS states in type IIB/Y are obtained by wrapping $D3$-branes on special Lagrangian cycles (SLAGs) of Y. More precisely, they are realized as elements in the Fukaya category $Fuk(Y)$.

BPS spectrum and homological mirror symmetry II

- These algebras are graded by the charge vector $\gamma \in H_{\rm even}(X, \mathbb{Z})$ in type IIA, or $\gamma \in H_3(Y, \mathbb{Z})$ in type IIB (more accurately, $\gamma \in K(X)$)

Minasian, Moore, …

- Each of these derived categories are endowed with a stability condition, determined by a choice of point in $\mathcal{M}_V$, which allows to decide which D-brane configurations are stable.
- The number of such configurations (counted with sign) defines the generalized Donaldson-Thomas invariant $\Omega(\gamma, t)$. It is a locally constant function on $\mathcal{M}_V$. It can jump on certain codimension one walls in $\mathcal{M}_V$, known as lines of marginal stability (LMS), according to certain (recently established) wall-crossing formulae.

Bridgeland; Joyce Son; Kontsevich Soibelman...

BPS spectrum and homological mirror symmetry III

- For D-brane charge $\gamma = [X] \oplus 0 \oplus q_a \gamma^a \oplus 2J[pt] \in H_{\rm even}(X)$ in type IIA, and t in a suitable domain, $\Omega(\gamma, t) = N_{DT}(q, 2J)$. In general, it provides a non-Abelian generalization of the Donaldson-Thomas invariants.
- The homological mirror symmetry conjecture states that $DCoh(X, t) = Fuk(Y, t)$ as an isomorphism of triangulated categories with a stability condition. In particular, the generalized Donaldson-Thomas invariants must agree.

Kontsevich

- Physically, $\Omega(\gamma, t)$ arises as the Witten index of some SUSY quantum mechanical system (quiver gauge theory) describing the microscopic dynamics of open strings in a D-brane background,

$$\Omega(\gamma, t) = \mathrm{Tr}_{\mathcal{H}_{BPS(\gamma,t)}}(-1)^F$$

- BPS states have an alternative macroscopic description as certain BPS solutions in $N = 2$ supergravity. Some of them are spherically symmetric black hole solutions of Reissner-Nordström type. Others are molecule-like bound states of several BPS black holes, whose relative distances depend on the value of the VM moduli at spatial infinity.

Microscopics and Macroscopics of BPS states II

- The Bekenstein-Hawking formula gives a powerful prediction for the growth of the BPS degeneracies [A=total horizon area]

$$\Omega(\gamma) \sim e^{\frac{1}{4}A(\gamma)} \qquad \text{as} \qquad |\gamma| \to \infty,$$

- The LMS is characterized by one of the relative distances going to infinity, i.e the bound state decaying into multi-particle states. This gives a macroscopic prediction for the wall-crossing formula obeyed by $\Omega(\gamma, t)$. E.g, for primitive vectors γ_1, γ_2,

$$\Delta\Omega(\gamma_1 + \gamma_2) = (-1)^{\langle\gamma_1,\gamma_2\rangle} \langle\gamma_1, \gamma_2\rangle \, \Omega(\gamma_1)\, \Omega(\gamma_2)$$

Denef Moore

Outline

Perturbative hypermultiplet moduli space I

- At weak coupling g_s, the story with $\mathcal{M}_H$ runs very similar to $\mathcal{M}_V$. However, unlike $\mathcal{M}_V$, $\mathcal{M}_H$ receives non-perturbative corrections from D-brane and NS5-brane instantons. Thus, it combines Gromov-Witten theory, generalized Donaldson-Thomas theory and presumably new math/physics related to NS5-branes.
- $\mathcal{M}_H$ is a quaternion-Kähler space of real dimension $4(h_{1,2}(X)+1)$. Despite the name, $\mathcal{M}_V$ is not Kähler, and carries no (globally defined) complex structure.
- In type IIA/X, $\mathcal{M}_H \equiv \mathcal{QK}_C(X)$ parametrizes the complex structure of X, together with the string coupling constant g_s, the RR 3-form $\mathcal{R} \in \mathrm{Jac}(X) \equiv H^3(X,\mathbb{R})/H^3(X,\mathbb{Z})$ and the NS-axion $\sigma \in S^1$.

Perturbative hypermultiplet moduli space II

- In the limit $g_s \to 0$, the QK metric on $\mathcal{M}_H$ looks like

$$\mathrm{d}s^2_{\mathcal{M}_H} = \left(\frac{\mathrm{d}g_s}{g_s}\right)^2 + \mathrm{d}s^2_{\mathcal{M}_C} + g_s^2\, \mathrm{d}s^2_{\mathrm{Jac}} + g_s^4\, (\mathrm{d}\sigma + \mathcal{A})^2$$

  where $\mathrm{d}s^2_{\mathcal{M}_C}$ is the PSK metric on the moduli space of complex structures, the same as the VM moduli space in type IIB, and $\mathcal{A}$ is a connection on the circle bundle S^1_σ, with 1st Chern class $\mathrm{d}\mathcal{A} \propto \omega_{\mathrm{Jac}}$. This is known as the "c-map" or "semi-flat" metric.

  Ceccoti Ferrara Girardello; Ferrara Sabharwal

- The effect of the one-loop correction in string theory is (roughly) to shift $g_s^2 \to g_s^2 + \chi$ and $\omega_{\mathrm{Jac}} \to \omega_{\mathrm{Jac}} + \chi\, \omega_{\mathcal{M}_C}$. As a result, the metric has a curvature singularity at $g_s^2 \sim \chi$.

  Antoniadis Minasian Theisen Vanhove; Günther Herrmann Louis, ...

Perturbative hypermultiplet moduli space III

- No perturbative corrections to $\mathrm{d}s_{\mathcal{M}_H}$ are expected beyond one-loop, since they would ruin the quantization of $c_1(\mathcal{A})$.
- Instanton corrections from Euclidean D2-branes wrapping SLAGs are expected to break the translational isometries along $\mathrm{Jac}(X)$ at order e^{-1/g_s}.
- Instanton corrections from Euclidean NS5-branes wrapping X are expected to break the translational isometry along S^1_σ at order e^{-1/g_s^2}.
- The challenge is to find the exact quantum corrected QK metric on $\mathcal{M}_H$. Unfortunately, type II string perturbation theory does not tell us immediately how...

Vectors meet hypers... I

- The occurrence of $\mathcal{M}_V^{IIB}$ in the limit $g_s \to 0$ of $\mathcal{M}_H^{IIB}$ is not coincidental. Consider type IIB string theory compactified on the same CY X, further reduced on a circle of radius R to $D = 3$.
- In $D = 3$, all 1-forms are Hodge dual to 0-forms, $\mathrm{d}A_1 = *\mathrm{d}A_0$. The moduli space is now a product of two quaternion-Kähler manifolds $\mathcal{M}_H^{IIB} \times \mathcal{M}_V^{IIB'}$ of real dimensions $4h_{12}(X) + 4$ and $4h_{11}(X) + 4$. The first is just the HM moduli space in $D = 4$.
- The second factor describes the VM moduli t^a in $D = 4$, the radius R, the holonomy and hodge duals $\mathcal{R} \in \mathrm{Jac}(X)$ of the 1-forms, and the Hodge dual σ of the Kaluza-Klein connection g_{i4}. At large radius R, the QK metric looks like

$$\mathrm{d}s^2_{\mathcal{M}_V^{IIB'}} = \left(\frac{\mathrm{d}R}{R}\right)^2 + \mathrm{d}s^2_{\mathcal{M}_C} + \frac{1}{R^2}\,\mathrm{d}s^2_{\mathrm{Jac}} + \frac{1}{R^4}\left(\mathrm{d}\sigma + \mathcal{A}\right)^2$$

Vectors meet hypers... II

- This is looks the same as $\mathcal{M}_H^{IIA}$! In fact, T-duality along the circle identifies $\mathcal{M}_V^{IIB'} = \mathcal{M}_H^{IIB}$, $g_s^A = 1/R_B$. Euclidean D2-branes on $H_3(X)$ are mapped to Euclidean D3-branes on $H_4(X \times S_1)$, i.e. black holes in $D = 4$! NS5-branes on X are mapped to Taub-NUT instantons (or Kaluza-Klein monopoles) on $X \times S_1$.
- Similarly $\mathcal{M}_V^{IIA'} = \mathcal{M}_H^{IIB}$. In addition, mirror symmetry identifies $M_H^{IIA}(X)$ with $M_H^{IIB}(Y)$. To sum up, for a given CY threefold X, type II string theory associates two QK manifolds $\mathcal{QK}_K(X)$ and $\mathcal{QK}_C(X)$, such that the moduli spaces in $D = 3$ are given by

	$IIA/X \times S^1$	$IIB/X \times S^1$
$\mathcal{M}'_V \times \mathcal{M}_H$	$\mathcal{QK}_K(X) \times \mathcal{QK}_C(X)$	$\mathcal{QK}_C(X) \times \mathcal{QK}_K(X)$

Constraints on the exact HM moduli space I

- In the weak coupling limit $g_s \to 0$ (or $R \to \infty$), the QK metric must reduce to the semi-flat metric;
- D-instanton effects should be weighted by the generalized DT invariants $\Omega(\gamma, t)$ (up to multi-covering effects);
- The metric should be smooth and complete; in particular, continuous across LMS, and regular at $g_s^2 \sim \chi$;
- Under mirror symmetry, $\mathcal{M}^K(X) = \mathcal{M}^C(Y)$;

Constraints on the exact HM moduli space II

- $\mathcal{M}_K(X)$ should have an isometric action of $SL(2,\mathbb{Z})$, inherited from the 10D S-duality symmetry of type $IIB/X \times S^1$, or equivalently from the global diffeomorphisms of T^2 in M-theory on $X \times T^2$.
- There are also reasons to expect an isometric action of $SL(3,\mathbb{Z})$, coming from 4D S-duality or Ehlers symmetry, or of a Picard subgroup $SU(2,1,\mathbb{Z}[\sqrt{-d}])$ for certain rigid CY three-folds with complex multiplication.

BP Persson; Bao Kleinschmidt Nilsson Persson BP

- When X admits a K3-fibration with a global section, one could in principle use heterotic-type II duality to compute the metric on $\mathcal{M}(X)$ using $(0,4)$ SCFT techniques. This is promising, but little has been accomplished so far.

A word on the rigid limit I

- In the limit where X becomes singular, it is sometimes possible to decouple gravity and describe the low energy physics in terms of an ordinary field theory with $N = 2$ "rigid" supersymmetries.
- This is in particular so when X develops an A_{N-1} singularity, fibered over a Riemann surface Σ. The D2-branes wrapped on vanishing cycles lead to massless gauge bosons, described by $SU(N)$ $N = 2$ Super-Yang-Mills.
- The SK metric on the Coulomb branch (where the gauge group is broken to $U(1)^{N-1}$) is described again by a prepotential $F(X)$ (no longer homogeneous), which can be computed from period integrals on the Seiberg-Witten curve Σ.

A word on the rigid limit II

- The BPS spectrum exhibit similar chamber dependence and lines of marginal stability as in the SUGRA case.

Bilal Ferrari, ...

- Upon reduction on a circle, the VM moduli space is enhanced to a hyperkähler manifold. When $R \to \infty$, it reduces to the 'rigid c-map' of the Coulomb branch. In addition there are $O(e^{-R})$ exponential corrections from BPS monopoles winding around the circle. The wall-crossing formula ensures that the HK metric is smooth across the LMS. The HK metric and BPS spectrum can be computed using integrable model techniques (Hitchin system).

Gaiotto Moore Neitzke

- In contrast to SUGRA, there are no $O(e^{-R^2})$ corrections, and the instanton sum converges.

Outline

QK geometry and contact geometry I

- Recall that a Riemannian manifold of real dimension $4n$ is quaternion-Kähler if its holonomy group is (exactly) $Sp(n) \times Sp(1)$. $\mathcal{M}$ is then Einstein. SUGRA requires negative scalar curvature. Let $\vec{p}$ be the $Sp(1)$ part of the Levi-Civita connection, $\mathrm{d}\vec{p} + \vec{p} \wedge \vec{p} = \frac{\nu}{2}\vec{\omega}$ the quaternionic 2-forms.
- $\mathcal{M}$ does not admit a (global) complex structure. Instead, it is more convenient to study its twistor space $\mathcal{Z}$. This is a complex contact manifold of real dimension $4n+2$, endowed with a (non-holomorphic) projection $\pi : \mathcal{Z} \to \mathcal{M}$ with CP^1 fibers, and a real structure acting as the antipodal map on CP^1.

Salamon; Lebrun

QK geometry and contact geometry II

- Explicitly, the complex contact structure on $\mathcal{M}$ is given by the kernel of the $(1,0)$-form Dz (which transforms homogeneously under $Sp(1) = SU(2)$ frame rotations)

$$Dz = \mathrm{d}z + p_+ - \mathrm{i}p_3 z + p_- z^2$$

Moreover, $\mathcal{M}$ carries a Kahler-Einstein metric

$$\mathrm{d}s^2_{\mathcal{Z}} = \frac{|Dz|^2}{(1+z\bar{z})^2} + \frac{\nu}{4}\mathrm{d}s^2_{\mathcal{M}}$$

- Locally, there exists a "contact potential" $\Phi(x^\mu, z)$ and Darboux complex coordinates $\alpha, \xi, \tilde{\xi}$ such that

$$\mathcal{X} = 2\, e^{\Phi} \frac{Dz}{z} = \mathrm{d}\alpha + \xi^\Lambda \mathrm{d}\tilde{\xi}_\Lambda$$

Φ provides a Kähler potential K on $\mathcal{Z}$ via $e^K = (1+z\bar{z})e^{Re(\Phi)}/|z|$.

QK geometry and contact geometry III

- The complex contact structure can be specified globally by providing contactomorphisms on the overlap of two Darboux coordinate patches. Those are conveniently specified by a Hamilton function $S^{[ij]}(\xi^\Lambda_{[i]}, \tilde\xi^{[j]}_\Lambda, \alpha^{[j]})$:

$$\xi^\Lambda_{[j]} = f_{ij}^{-2}\, \partial_{\tilde\xi^{[j]}_\Lambda} S^{[ij]}\,, \qquad \tilde\xi^{[i]}_\Lambda = \partial_{\xi^\Lambda_{[i]}} S^{[ij]}\,,$$
$$\alpha^{[i]} = S^{[ij]} - \xi^\Lambda_{[i]} \partial_{\xi^\Lambda_{[i]}} S^{[ij]}\,, \qquad e^{\Phi_{[i]}} = f_{ij}^2\, e^{\Phi_{[j]}}\,,$$

  where $f_{ij}^2 \equiv \partial_{\alpha^{[j]}} S^{[ij]} = \mathcal{X}^{[i]}/\mathcal{X}^{[j]}$.

- $S^{[ij]}$ are subject to consistency conditions $S^{[ijk]}$, gauge equivalence under local contact transformations $S^{[i]}$, and reality constraints.

QK geometry and contact geometry IV

- For generic choices of $S^{[ij]}$, the moduli space of solutions of the above gluing conditions, regular in each patch, is finite dimensional, and equal to $\mathcal{M}$ itself.
- On each patch U_i, $u_m^{[i]} = (\xi^\Lambda_{[i]}, \tilde\xi^{[i]}_\Lambda, \alpha^{[i]})$ admit a Taylor expansion in z around ζ_i, whose coefficients are functions on $\mathcal{M}$. The functions $u_m^{[i]}(z, x^\mu)$ parametrize the "twistor line" over $x^\mu \in \mathcal{M}$.
- The metric on $\mathcal{M}$ can be obtained by expanding $\mathcal{X}^{[i]}$ and $\mathrm{d}u_m^{[i]}$ around z_i, extracting the $SU(2)$ connection $\vec{p}$ and a basis of $(1,0)$ forms on $\mathcal{M}$ in almost complex structure $J(z_i)$, and using $\mathrm{d}\vec{p} + \frac{1}{2}\,\vec{p}\times\vec{p} = \frac{\nu}{2}\,\vec{\omega}$.
- Deformations of $\mathcal{M}$ correspond to deformations of $S^{[ij]}$, so are parametrized by $H^1(\mathcal{Z}, \mathcal{O}(2))$.

Lebrun, Salamon

QK geometry and contact geometry V

- Any (infinitesimal) isometry κ of $\mathcal{M}$ lifts to a holomorphic isometry $\kappa_{\mathcal{Z}}$ of $\mathcal{Z}$. The moment map construction provides an element of $H^0(\mathcal{Z}, \mathcal{O}(2))$, given locally by holomorphic functions

$$\mu_\kappa = \kappa_{\mathcal{Z}} \cdot \mathcal{X} = e^{\Phi} \left(\mu_+ z^{-1} - \mathrm{i}\mu_3 + \mu_- z \right) .$$

Galicki

The moment map of the Lie bracket $[\kappa_1, \kappa_2]$ is the contact-Poisson bracket $\{\mu_{\kappa_1}, \mu_{\kappa_2}\}_{PB}$. The zeros of μ canonically associate a (local) complex structure J_κ to κ.

- Toric QK manifolds are those which admit $d+1$ commuting isometries. In this case, one can choose $\mu_{[i]}$ as the position coordinates. The transition functions must then take the form

$$S^{[ij]} = \alpha^{[j]} + \xi^{\Lambda}_{[i]} \tilde{\xi}^{[j]}_{\Lambda} - H^{[ij]} ,$$

where $H^{[ij]}$ depends on $\xi^{\Lambda}_{[i]}$ only.

QK geometry and contact geometry VI

- More generally, one can consider "nearly toric QK", where $H^{[ij]}$ is a general function but its derivatives wrt to $\tilde{\xi}_\Lambda^{[i]}, \alpha^{[i]}$ are taken to be infinitesimal. For one unbroken isometry κ, $\partial_{\alpha^{[i]}} H^{[ij]} = 0$.
- The twistor lines can then be obtained by Penrose-type integrals, e.g. (in case with one isometry, no "anomalous dimensions")

$$\xi_{[i]}^\Lambda = \zeta^\Lambda + \frac{Y^\Lambda}{z} - z\bar{Y}^\Lambda - \frac{1}{2}\sum_j \oint_{C_j} \frac{\mathrm{d}z'}{2\pi\mathrm{i}z'} \frac{z'+z}{z'-z} \partial_{\tilde{\xi}_\Lambda^{[j]}} H^{[+j]}(z')$$

$$e^{\Phi_{[i]}} = \frac{1}{4}\sum_j \oint_{C_j} \frac{\mathrm{d}z'}{2\pi\mathrm{i}z'} \left(z'^{-1}Y^\Lambda - z'\bar{Y}^\Lambda\right) \partial_{\xi_{[j]}^\Lambda} H^{[+j]}(\xi(z'), \tilde{\xi}(z'))$$

The locus $z = 0$ defines the canonical complex structure J_κ.

Outline

The perturbative hypermultiplet moduli space I

- Let us now return to the HM moduli space $\mathcal{M}_H$ in type IIA compactified on X. For simplicity, assume $\chi(X) = 0$. In string perturbation theory, $\mathcal{M}_H^{\rm pert} \sim \rm c - map(\mathcal{M}_V^{IIB})$.
- The twistor space is governed by the Hamilton functions

$$H^{[0+]}_{\rm pert} = \frac{i}{2} F(\xi^\Lambda) \,, \quad H^{[0-]}_{\rm tree} = \frac{i}{2} \bar F(\xi^\Lambda)$$

Roček Vafa Vandoren

- As a result, the twistor lines are given [upon defining $\tilde\xi_\Lambda \equiv -2{\rm i}\tilde\xi^{[0]}_\Lambda, \alpha \equiv 4{\rm i}\alpha^{[0]} + 2{\rm i}\tilde\xi^{[0]}_\Lambda \xi^\Lambda$, $W(z) \equiv F_\Lambda \zeta^\Lambda - X^\Lambda \tilde\zeta_\Lambda$] by

$$\begin{aligned} \xi^\Lambda &= \zeta^\Lambda + \left(z^{-1} X^\Lambda - z\, \bar X^\Lambda\right) / g_s^2 \,, \\ \tilde\xi_\Lambda &= \tilde\zeta_\Lambda + \left(z^{-1} F_\Lambda - z\, \bar F_\Lambda\right) / g_s^2 \,, \\ \alpha &= \sigma + \left(z^{-1} W - z\, \bar W\right) / g_s^2 \,, \end{aligned}$$

Neitzke BP Vandoren; Alexandrov; APSV

Generalized Mirror Map I

- Using mirror symmetry, the perturbative contact potential may be written in terms of the GW invariants of Y [here $\tau_2 = 1/g_s$],

$$e^{\Phi} = \frac{\tau_2^2}{2} V + \frac{\tau_2^2}{4(2\pi)^3} \sum_{q_a \gamma^a \in H_2^+(Y)} n_{0,q_a} \mathrm{Re}\left[\mathrm{Li}_3\left(\mathrm{e}^{q}\right) + 2\pi q_a t^a \, \mathrm{Li}_2\left(\mathrm{e}^{q}\right) \right]$$

while the RR multiform $\zeta^\Lambda, \tilde\zeta_\Lambda$ and NS-axion σ are related to type IIB variables $\tau_1, c^a, c_a, c_0, \psi$ by the "generalized mirror map"

$$\zeta^0 = \tau_1 \,, \qquad \zeta^a = -(c^a - \tau_1 b^a) \,,$$

$$\tilde\zeta_a = c_a + \frac{1}{2} \kappa_{abc} \, b^b (c^c - \tau_1 b^c) \,, \quad \tilde\zeta_0 = \, c_0 - \frac{1}{6} \kappa_{abc} \, b^a b^b (c^c - \tau_1 b^c) \,,$$

$$\sigma = -2(\psi + \frac{1}{2} \tau_1 c_0) + c_a (c^a - \tau_1 b^a) - \frac{1}{6} \kappa_{abc} \, b^a c^b (c^c - \tau_1 b^c) \,.$$

Gunther Herrmann Louis; Berkooz BP; APSV

S-duality and symplectic covariance I

- In the weak coupling, large IIB volume limit, $\mathcal{M}_H$ admits an isometric action of $SL(2,\mathbb{R})$

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}\,, \qquad j^a \mapsto j^a |c\tau + d|\,, \qquad c_a \mapsto c_a\,,$$
$$\begin{pmatrix} c^a \\ b^a \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} c^a \\ b^a \end{pmatrix}, \qquad \begin{pmatrix} c_0 \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} c_0 \\ \psi \end{pmatrix}$$

- This can be lifted to a holomorphic action on Z,

$$\xi^0 \mapsto \frac{a\xi^0 + b}{c\xi^0 + d}\,, \qquad \xi^a \mapsto \frac{\xi^a}{c\xi^0 + d}\,, \qquad \ldots$$

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S-duality and symplectic covariance II

- The contact potential $e^{\Phi} = \frac{\tau_2^2}{2} V(j^a)$, though not invariant, transforms so that $K_{\mathcal{Z}}$ undergoes a Kähler transformation,

$$e^{\Phi} \mapsto \frac{e^{\Phi}}{|c\tau + d|}\,, \quad K_{\mathcal{Z}} \mapsto K_{\mathcal{Z}} - \log(|c\xi^0 + d|)\,, \quad \mathcal{X}^{[i]} \to \frac{\mathcal{X}^{[i]}}{c\xi^0 + d}$$

- The worldsheet instanton corrections break $SL(2,\mathbb{R})$ continuous S-duality. A discrete subgroup $SL(2,\mathbb{Z})$ can be restored by summing over images:

$$\mathrm{Li}_k(e^{2\pi \mathrm{i} q_a z^a}) \to \sum_{m,n}' \frac{\tau_2^{k/2}}{|m\tau + n|^k} e^{-S_{m,n,q}}\,,$$

where $S_{m,n,q} = 2\pi q_a |m\tau + n|\, t^a - 2\pi \mathrm{i} q_a (mc^a + nb^a)$ is the action of a (m,n)-string wrapped on $q_a \gamma^a$.

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S-duality and symplectic covariance III

- After Poisson resummation on $n \to q_0$, we get a sum over D(-1)-D1 bound states, $e^{\Phi} = \cdots +$

$$\frac{\tau_2}{8\pi^2} \sum_{\substack{q_0 \in \mathbb{Z} \\ q_a\gamma^a \in H_2^+(Y)}} n^{(0)}_{q_a} \sum_{m=1}^{\infty} \frac{|q_\Lambda X^\Lambda|}{m} \cos\left(2\pi m\, q_\Lambda \zeta^\Lambda\right) K_1\left(2\pi m\, |q_\Lambda X^\Lambda| \tau_2\right)$$

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- Going back to type IIA variables, these are interpreted as Euclidean $D2$ wrapped on SLAG in a Lagrangian subspace of $H_3(X, \mathbb{Z})$ (A-cycles only). These effects correct the mirror map into

$$\tilde{\zeta}_a = \tilde{\zeta}_a^{(0)} + \frac{1}{8\pi^2} \sum_{q_a} n_{0,q} \sum_{n \in \mathbb{Z}, m \neq 0} \frac{m\tau_1 + n}{m|m\tau + n|^2} e^{-S_{m,n,q}}, \ldots$$

Alexandrov Saueressig

S-duality and symplectic covariance IV

- In the "one instanton" approximation, the contributions of B-cycles can be restored by symplectic invariance:

$$e^{\Phi} = \cdots + \frac{\tau_2}{8\pi^2} \sum_{\gamma} n_\gamma \sum_{m=1}^{\infty} \frac{|W_\gamma|}{m} \cos\left(2\pi m \Theta_\gamma\right) K_1\left(2\pi m |W_\gamma|\right)$$

$$W_\gamma \equiv \frac{1}{2}\tau_2 \left(q_\Lambda X^\Lambda - p^\Lambda F_\Lambda \right) , \quad \Theta_\gamma \equiv q_\Lambda \zeta^\Lambda - p^\Lambda \tilde{\zeta}_\Lambda$$

- At this point, n_γ just parametrize the allowed deformations. However, their behavior under wall-crossing and general expectations from T-duality suggest that $n_\gamma = \Omega(\gamma, t)$, the generalized DT invariants.

The hypermultiplet twistor space I

- The contact structure on the twistor space can be obtained by inserting an elementary symplectomorphism generated by

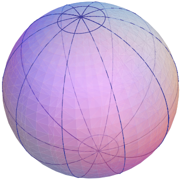
$$S_\gamma^{[ij]}(\xi^\Lambda_{[i]}, \tilde\xi^{[j]}_\Lambda, \alpha^{[j]}) = \alpha^{[j]} + \xi^\Lambda_{[i]}\, \tilde\xi^{[j]}_\Lambda + \frac{\mathrm{i}}{2(2\pi)^2}\, n_\gamma\, \mathrm{Li}_2\left(\mathcal{X}_\gamma\right)\,.$$

Gaiotto Moore Neitzke

across the "BPS ray" $\ell(\gamma)$,

$$\ell(\gamma) = \{z:\ \pm W_\gamma/z \in \mathrm{i}\mathbb{R}^-\}\ ,$$

$$\mathcal{X}_\gamma = e^{-2\pi\mathrm{i}(q_\Lambda \xi^\Lambda_{[i]} + 2\mathrm{i} p^\Lambda \tilde\xi^{[j]}_\Lambda)}$$



- As $t \in \mathcal{M}_V$ is varied, the BPS rays may cross, and the invariants n_γ should transform so as to leave the contact structure intact.

The hypermultiplet twistor space II

- BPS rays $\ell(\gamma_1)$ and $\ell(\gamma_2)$ cross at lines of marginal stability. The wall crossing formula

$$\prod^{\curvearrowleft}_{\substack{\gamma=n\gamma_1+m\gamma_2\\ m>0,n>0}} U_\gamma^{n^-(\gamma)} = \prod^{\curvearrowright}_{\substack{\gamma=n\gamma_1+m\gamma_2\\ m>0,n>0}} U_\gamma^{n^+(\gamma)} ,$$

  ensures that the consistency of the twistor space across the LMS.

  Gaiotto Neitzke Moore; Kontsevich Soibelman

- The metric is regular across the LMS. Physically, single instanton contributions on one side of the wall get replaced by multi-instanton configurations on the other side.

Black holes, Taub-NUT instantons and NS5-branes I

- If indeed $n_{\gamma,t} = \Omega(\gamma, t) \sim e^{\frac{1}{4}A(\gamma)}$, the instanton series is divergent, and must be treated as an asymptotic series. Its accuracy can be estimated by Borel type techniques. Schematically,

$$\sum_Q e^{Q^2 - Q/g_s} \sim e^{-1/g_s^2}$$

Thus NS5-brane or KK-monopoles are expected to play a crucial role in regulating the black hole sum.

BP Vandoren

Black holes, Taub-NUT instantons and NS5-branes II

- In contrast to D-instantons, NS5-brane instantons should induce genuine contact transformations, with $S^{[ij]} \propto e^{ik\alpha^{[ij]}} F_k(\xi, \tilde{\xi})$.
- For gauge invariance, F_k must be a holomorphic section of the Theta line bundle over $\mathrm{Jac}(X)$. This seems to fit with known facts about the NS5-brane partition function, and about the topological string amplitude !

Witten; Freed Moore Belov; Dijkgraaf Verlinde Vonk, ...

- One may in principle determine the NS5 instantons by $SL(2, \mathbb{Z})$ duality from the D5-instantons. Automorphy under $SL(3, \mathbb{Z})$ provides a short cut.
- There are indications that the motivic DT invariants and the quantum dilogarithm should play an important role in this story, although it is unclear yet how.

Kontsevich Soibelman; Dimofte Gukov, ...

Conclusion I

- Determining the exact HM metric is hard, but (hopefully) not impossible. Twistor methods are essential, but can still be improved (completeness, discrete symmetries...)
- In some highly symmetric cases (e.g. Enriques or Borcea-Voisin CY), one may hope that automorphy will fix the hypermultiplet metric exactly, giving access to new CY invariants. Heterotic/type II duality may also be a very powerful approach.
- The metric on $\mathcal{M}_H$ offers a very convenient packaging of the degeneracies of 4D BPS black holes. Divergences of the BH partition function should be resolved by NS5 or TN-instantons.
- It seems that higher derivative $\tilde{F}_g$-type corrections to the hypers should be governed by a one-parameter generalization of the topological string amplitude, which mixes A and B-model data. The way to non-perturbative topological string theory ?