Quaternionic Geometry, instanton corrections and wall crossing

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Goal: describe some recent constructions of quaternionic (HK/QK) metrics on certain complex families of Abelian varieties, motivated by the physics of $\mathcal{N}=2$ supersymmetric field and string theories: $T \longrightarrow \mathcal{M} \longrightarrow \mathcal{B}$

The metric depends on certain "BPS invariants", which are constant, integer valued functions on $\mathcal{B}$, away from certain codimension one "walls".
The smoothness of the metric across the wall is equivalent to the Kontsevich - Soibelman wall-crossing formula.
, formulated in twistor space,

Main references:

Gaiotto Moore Neitzke 0807.4723 for HK

Alexandrov BP Saueressig Vandoren 0812.4219 for QK

Kontsevich Soibelman 0910.4315 } for wall-crossing
Joyce - Song 0810.5645 }

More refs are provided in my Trieste lectures (in progress, intended for physicists) available on www.lpthe.jussieu.fr/~pioline/seminars.html

1. General set-up and goals
   - Some physically interesting families of Abelian varieties
   - The c-map construction of the large radius metric
   - BPS invariants, wall crossing and instanton corrections

2. Hyperkähler metrics [field theory]
   - Twistor techniques and Legendre transform construction
   - The rigid c-map in twistor space
   - Electric instantons and the Ooguri-Vafa metric
   - The GMN construction

3. Quaternion-Kähler metrics [string theory]
   - Twistor techniques
   - The local c-map in twistor space
   - D-instantons
   - S-duality and NS5 instantons

lecture 1: General set-up and goals

1.1 Some physically interesting families of Abelian varieties arise as follows: let $\mathcal{B}$ a complex manifold, $\dim_{\mathbb{C}} \mathcal{B} = r$

Γ a rank-$2r$ lattice equipped with an integer symplectic pairing

$$\langle \gamma, \gamma' \rangle = - \langle \gamma', \gamma \rangle \in \mathbb{Z}$$

let $Z : \mathcal{B} \longrightarrow \mathrm{Hom}(\Gamma, \mathbb{C})$ be a "central charge function"

such that $\mathrm{Im}\, \mathcal{B}$ is Lagrangian in the complex symplectic space $\mathrm{Hom}(\Gamma, \mathbb{C}) = \Gamma^* \otimes \mathbb{C}$

In coordinates: $\Gamma = \Gamma_e \oplus \Gamma_m$ $\qquad \Lambda = 1..r$

$$= \mathrm{Span}(A^\Lambda) \oplus \mathrm{Span}(B_\Lambda)$$

$$\langle A^\Lambda, A^\Sigma \rangle = \langle B_\Lambda, B_\Sigma \rangle = 0, \quad \langle A^\Lambda, B_\Sigma \rangle = \delta^\Lambda_\Sigma$$

$$\gamma = q_\Lambda A^\Lambda - p^\Lambda B_\Lambda \qquad p, q \in \mathbb{Z}$$

u coordinates on $\mathcal{B}$

$$Z(\gamma, u) = q_\Lambda X^\Lambda - p^\Lambda F_\Lambda \quad \text{where} \quad \begin{cases} X^\Lambda = Z(A^\Lambda, u) \\ F_\Lambda = Z(B_\Lambda, u) \end{cases}$$

The cx symplectic form on $\mathrm{Hom}(\Gamma, \mathbb{C})$ is

$$dX^\Lambda \wedge dF_\Lambda$$

so the Lagrangian condition means that locally (possibly after an integer symplectic rotation)

$$F_\Lambda(u) = \frac{\partial F(X)}{\partial X^\Lambda}(u) \qquad F \text{ "prepotential"}$$

Rk Z may have monodromies; Γ_u is really a local system on $\mathcal{B}$

The space $\mathrm{Hom}(\Gamma, \mathbb{C})$ also carries a real symplectic form

$$\frac{1}{2}\left(dX^\Lambda \wedge d\bar{F}_\Lambda - d\bar{X}^\Lambda \wedge dF_\Lambda\right)$$

This pulls back to a symplectic two-form on $\mathcal{B}$

$$\omega_{SK} = i \,\mathrm{Im}\,\tau_{\Lambda\Sigma}\, dX^\Lambda d\bar{X}^\Sigma \qquad \text{where} \qquad \tau_{\Lambda\Sigma} \equiv \partial_{X^\Lambda}\partial_{X^\Sigma} F$$

We assume for now that $\boxed{\mathrm{Im}\,\tau_{\Lambda\Sigma} > 0}$

so that $\mathcal{B}$ is endowed with a <u>special Kähler</u> metric

$$ds^2_{\mathcal{B}} = \frac{1}{2}\mathrm{Im}\,\tau_{\Lambda\Sigma}\, \partial_{u^a} X^\Lambda\, \partial_{\bar{u}^{\bar{b}}} \bar{X}^\Sigma\, du^a d\bar{u}^{\bar{b}}$$

We now consider, over each point $u \in \mathcal{B}$, the Abelian variety

$$T_u = \Gamma^* \otimes \mathbb{R}/2\pi\mathbb{Z} \quad , \quad \text{coordinates } C \equiv \begin{pmatrix}\zeta^\Lambda \\ \tilde{\zeta}_\Lambda\end{pmatrix} \in \mathbb{R}/2\pi\mathbb{Z}$$

with Kähler metric

$$ds^2_T = \frac{1}{2}\mathrm{Im}\,\tau^{\Lambda\Sigma}\left(d\tilde{\zeta}_\Lambda - \tau_{\Lambda\Lambda'}\, d\zeta^{\Lambda'}\right)\left(d\tilde{\zeta}_\Sigma - \bar{\tau}_{\Sigma\Sigma'}\, d\zeta^{\Sigma'}\right)$$

The family of Abelian varieties $\quad T_u \longrightarrow \mathcal{M} \longrightarrow \mathcal{B}$

carries a family of HK metrics [locally isometric]

$$ds^2_{\mathcal{M}}(R) = ds^2_{\mathcal{B}} + \frac{1}{R^2}\, ds^2_T$$

with complex symplectic form

$$\omega_+ = dX^\Lambda \wedge dW_\Lambda , \qquad W_\Lambda = i\left(\tilde{\zeta}_\Lambda - \tau_{\Lambda\Lambda'}\zeta^{\Lambda'}\right)$$

and Kähler potential

$$K = i\left(X^\Lambda \bar{F}_\Lambda - \bar{X}^\Lambda F_\Lambda\right) + \frac{1}{R^2}\left(W_\Lambda + \bar{W}_\Lambda\right) \mathrm{Im}\,\tau^{\Lambda\Sigma}\left(W_\Sigma + \bar{W}_\Sigma\right)$$

$\hookrightarrow$ the "rigid c-map" metric

(Cecotti Ferrara Girardello)

(Reminder: a Riemannian manifold $\mathcal{M}_{4r}$ is HK iff
it has reduced holonomy $Sp(r) \subset SO(4r)$

Equivalently, it admits a triplet of endo of tangent bundle J_i, $i=1,2,3$
such that

1) $$J_i J_j = -\delta_{ij} + \epsilon_{ijk} J_k$$

2) $$D J_i = 0$$

$\mathcal{M}$ is then Kähler wrt to each cx structure: ω_i Kähler forms

$\omega_+ = \omega_1 + i\omega_2$ is of type $(2,0)$ wrt to J_3,
and in fact $\bar\partial$-closed and symplectic
$\rightarrow$ $\omega_+ = dX^\Lambda \wedge dW_\Lambda$ locally in Darboux coordinates)

1.2 Physical example: Seiberg-Witten theories,
i.e. $\mathcal{N}=2$ super Yang-Mills theory in 4D.
G gauge group of rank r

$\mathcal{B}$ = moduli space of Coulomb vacua on $\mathbb{R}^4$

$\mathcal{M}$: moduli space of Coulomb vacua on $\mathbb{R}^3 \times S^1$

$$\zeta^\Lambda = \oint_{S_1} A^\Lambda \qquad\qquad \mathcal{A}^\Lambda, \mathcal{A}_\Lambda \text{ electric and magnetic}$$
$$\tilde\zeta_\Lambda = \oint_{S_1} \mathcal{A}_\Lambda \qquad\qquad \text{Maxwell fields in } D=4.$$

To determine the metric, SW introduce a family of genus-r
Riemann surfaces Σ_u, and a meromorphic differential λ
such that

$$\Gamma_u = H_1(\Sigma_u, \mathbb{Z})$$
$$Z(\gamma; u) = \oint_\gamma \lambda \qquad \mathcal{M} = \mathrm{Jac}(\Sigma_u)$$

$ds^2_{\mathcal{M}}(R)$ is then an accurate representation of the physics at finite
radius R (ie to all orders in $1/R$)

– At finite R however, one expects exponentially suppressed corrections of order

$$\exp\left(-2\pi R\, |Z(\gamma,u)| + 2\pi i \langle \gamma, C\rangle\right)$$

due to BPS states in $D=4$, winding around the circle, which will break translational isometries along T_u.

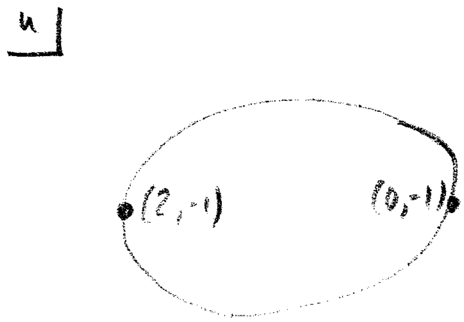
Recall that in $D=4$, the discrete spectrum contains special states which saturate the BPS bound

$$M \geq |Z(\gamma)|$$

The number of such states (counted with sign) $\Omega(\gamma;u)$ is an integer-valued function on $\mathcal{B}$, constant away from certain "walls of marginal stability", where the phase of the central charge associated to two charge vectors γ_1, γ_2 align:

$$W(\gamma_1,\gamma_2) = \{ u \in \mathcal{B} \;/\; \operatorname{Im} Z(\gamma_1,u)\,\bar{Z}(\gamma_2,u) = 0 \}$$

Eg, in $SU(2)$ Seiberg-Witten with no matter,



$\pm(2,0)$

$\pm(2n,1)$ $\quad n \in \mathbb{Z}_{\geq 0}$

$\pm(2n+2,-1)$

Across $W(\gamma_1,\gamma_2)$ a one-particle state with charge $\gamma = M\gamma_1 + N\gamma_2$ may turn into a multiparticle state with charges $\gamma_i = M_i\gamma_1 + N_i\gamma_2$, $\sum_i (M_i, N_i) = (M,N)$.

The Kontsevich-Soibelman / Joyce-Song formula allows to compute $\Delta\Omega(\gamma)$.

· Similarly, on $\mathbb{R}^3 \times S^1$, a single-instanton of charge γ may turn into a multi-instanton of charges $\{\gamma_i\}$, however the metric on $\mathcal{M}$ should be smooth (away from singularities of $\mathcal{B}$, where massless particles arise)

Q: can one construct a HK metric on $\mathcal{M}$, smooth across walls of marginal stability, which agrees with the c-map metric away from the walls?

Rk The same moduli space $\mathcal{M}$ can also be viewed as the moduli space of solutions to Hitchin's equations

$$F + R^2 [\varphi, \bar{\varphi}] = 0 \qquad\qquad SU(K)$$
$$\bar{\partial}\varphi + [\bar{A}, \varphi] = 0$$

on a curve C with singularities, such that the SW curve Σ is a K-fold covering of C, see GMN II

1.3 QK moduli spaces associated to CY 3-folds

Let X be a CY 3-fold family, together with a choice of holomorphic 3-form $\Omega_{3,0}$. $\quad [\, h_{3,0}(X) = 1 \,]$

$(X, \Omega_{3,0})$ is fibered over $\mathcal{B}$, $\quad \dim_{\mathbb{C}} \mathcal{B} = h_{1,2} + 1 = r$

Let $\Gamma = H_3(X, \mathbb{Z})$

$$Z(\gamma, u) = \oint_\gamma \Omega_{3,0} = q_\Lambda X^\Lambda - p^\Lambda F_\Lambda$$

The previous construction leads to a family of $2r$-dim tori

$$T_u = H^3(X, \mathbb{R}) / H^3(X, \mathbb{Z}) \quad : \text{intermediate Jacobian torus}$$

endowed with the Griffiths complex structure:

$$\begin{array}{ccccccc} H^{3,0} & \oplus & H^{2,1} & \oplus & H^{1,2} & \oplus & H^{0,3} \\ i & & i & & -i & & -i \end{array}$$

This varies holomorphically over $\mathcal{B}$, however the metric $\operatorname{Im} \tau_{\Lambda\Sigma}$ has indefinite signature $(1, h_{1,2})$

Moreover, the base $\mathcal{B}$ itself has indefinite signature $(1, h_{1,2})$, the negative direction corresponding to rescalings $\Omega_{3,0} \to e^t \Omega_{3,0}$.

To arrive at a positive definite metric (as required physically), let

$$\mathcal{B}_0 = \mathcal{B} / \mathbb{C}^* \qquad \dim_{\mathbb{C}} = h_{1,2}$$

and replace the Griffiths complex structure by the Weil cx structure:

$$\begin{array}{ccccccc} H^{3,0} & \oplus & H^{2,1} & \oplus & H^{1,2} & \oplus & H^{0,3} \\ i & & -i & & i & & -i \end{array}$$

with period matrix $\quad \mathcal{N}_{\Lambda\Sigma} = \bar{\tau}_{\Lambda\Sigma} + 2i \dfrac{[\operatorname{Im}\tau \cdot X]_\Lambda \, [\operatorname{Im}\tau \cdot X]_\Sigma}{[X \operatorname{Im}\tau X]}$

- The family $T_u \longrightarrow \mathcal{M}_0 \longrightarrow \mathcal{B}_0$ is the "Weil intermediate Jacobian" of X, $\dim_{\mathbb{R}} = 4h_{1,2}+2$

Finally, consider a bundle of punctured disks $\mathcal{D} \to \mathcal{M} \to \mathcal{M}_0$ such that the first Chern class of the circle bundle in $\mathcal{D}$ is equal to the Kähler form on T_u:

$$ds^2_{\mathcal{D}} = \frac{dR^2}{R^2} + \frac{1}{R^4}\left(d\sigma + \xi^\Lambda d\tilde{\xi}_\Lambda - \tilde{\xi}_\Lambda d\xi^\Lambda \right)^2$$

<u>Thm (Ferrara, Sabharwal, "local c-map")</u>

The following metric is quaternion-Kähler:

$$ds^2_{\mathcal{M}} = ds^2_{\mathcal{D}} + ds^2_{\mathcal{B}_0} + \frac{1}{R^2}\, ds^2_{T_u} \qquad \dim_{\mathbb{R}} = 4h_{1,2}+4$$

Physically, $ds^2_{\mathcal{M}}$ describes the VM moduli space of type IIB string theory compactified on $X \times S_1 \times \mathbb{R}^3$. Note that the radius R is now a coordinate on $\mathcal{M}$, as opposed to a parameter in the HK story.

Moreover, unlike the HK case, $ds^2_{\mathcal{M}}$ is only accurate to leading order in $1/R$. The "1-loop corrected metric" is a simple QK deformation of the above, entirely determined by the Euler number of X.

However, as in the HK case, one expects exponentially suppressed corrections, of two types:

- $\sim \exp\left(-2\pi R|Z(\gamma,u)| + 2\pi i\langle C,\gamma\rangle\right)$, from Slag 3-cycles in X
- $\sim \exp\left(-\pi|k|R^2 + i\pi k\sigma\right)$, $k\in\mathbb{Z}$, whose mathematical origin is unclear.

As in the HK case, the number $\Omega(\gamma, u)$ of SLag 3-cycles in homological class γ varies with $\mathcal{B}_0$, according to the KS wall-crossing formula.

Q1: Can one construct a QK metric on $\mathcal{M}$, as an asymptotic series in $\exp(-R)$, smooth across the walls of marginal stability?

The dependence of the e^{-R^2} effects on $\mathcal{B}_0$ is, on the other hand, unclear. Yet, we expect that modular properties will fix these effects;

Q2: Can one construct a QK metric on $\mathcal{M}$, smooth away from "massless singularities" in $\mathcal{M}$, carrying an isometric action of $SL(2,\mathbb{Z})$?

A special case is also worth mentioning: if $h_{1,2}(X) = 0$, X is rigid and $\mathcal{M}$ has real dim 4 ("universal hypermultiplet").

Q3: Can one answer Q2 when X is rigid?

Finally, there is a mirror symmetric version of the above construction, where $\mathcal{B}_0$ is the moduli space of complexified Kähler structures on $\hat{X}$, $\Gamma = H_{even}(X, \mathbb{Z})$, SLAG $\to$ coherent sheaves.

Q4 (quantum mirror symmetry) show

$$\mathcal{M}_{complex}(X) \simeq_{QK} \mathcal{M}_{symp}(\hat{X}) \quad \text{if } (X, \hat{X}) \text{ mirror pair}$$

Lect 2. Hyperkähler metrics on complex integrable systems

d'après Gaiotto Moore Neitzke 0807.4723

2.1. Reminder

$$\begin{tikzcd} T \arrow[r] & \mathcal{M} \arrow[d] \\ & \mathcal{B} \ni u^a \end{tikzcd}$$

$\in$

$C = \begin{pmatrix} \zeta^\Lambda \\ \zeta_\Lambda \end{pmatrix}$

Γ rank $2r$ symplectic lattice

$T = \Gamma^* \otimes \mathbb{R}/2\pi\mathbb{Z}$

$Z : \mathcal{B} \longrightarrow \Gamma^*$

central charge function

Im $\mathcal{B}$ complex Lagrangian

$$Z(\gamma, u) = q_\Lambda X^\Lambda - p^\Lambda F_\Lambda$$

$$\tau_{\Lambda\Sigma} = \frac{\partial F_\Lambda}{\partial X^\Sigma} = \frac{\partial^2 F}{\partial X^\Lambda \partial X^\Sigma} \quad \text{period matrix}$$

$$\operatorname{Im} \tau_{\Lambda\Sigma} > 0$$

Thm (Cecotti Ferrara Girardello, "rigid c-map")

$$K = i\left(X^\Lambda \bar{F}_\Lambda - \bar{X}^\Lambda F_\Lambda\right) + \frac{1}{R^2}\left(W_\Lambda + \bar{W}_\Lambda\right)\left[\operatorname{Im}\tau^{-1}\right]^{\Lambda\Sigma}\left(W_\Sigma + \bar{W}_\Sigma\right)$$

is a Kähler potential for a HK metric on $\mathcal{M}$, in complex structure where $\begin{cases} X^\Lambda(u) \\ W_\Lambda = i\left(\tilde{\zeta}_\Lambda - \tau_{\Lambda\Sigma}\zeta^\Sigma\right) \end{cases}$ are local complex coordinates

and $\omega_+ = dX^\Lambda \wedge dW_\Lambda$ is the complex symplectic form.

Our aim is to construct a new (better behaved) HK metric on $\mathcal{M}$, which incorporates $\mathcal{O}\left(\Omega(\gamma,t)\exp\left(-2\pi R\,|Z(\gamma)| + 2\pi i\langle C,\gamma\rangle\right)\right)$ corrections, where $\Omega(\gamma, t)$ are BPS invariants satisfying the KS wall crossing formula

2.2. Twistor techniques for HK manifolds

Recall Hitchin's theorem:

If $(\mathcal{M}, ds^2)$ is HK_{4r}, * $\exists$ holomorphic fibration

$$\mathcal{M} \to Z \xrightarrow{p} \mathbb{P}^1 \qquad t \in \mathbb{C} \cup \{\infty\}$$

such that $p^{-1}(t) = \mathcal{M}$ in complex structure

$$J(t) = \frac{1-t\bar{t}}{1+t\bar{t}} J^3 + \frac{(t+\bar{t})}{1+t\bar{t}} J_2 + \frac{i(t-\bar{t})}{1+t\bar{t}} J^1$$

* Z carries an "$\mathcal{O}(2)$ twisted complex symplectic structure"

$\omega^{[i]} \in \Lambda^2 Z|_{U_i}$ holomorphic, closed, non degenerate along fib.

$\{U_i\}$ open covering of $\mathbb{P}^1$, f_{ij}^2 transition functions of $\mathcal{O}(2)$ over $\mathbb{P}$

$\omega^{[i]}$ is a holomorphic symplectic form on $\mathcal{M}$ in cx structure $J(t)$

$\omega^{[i]} = f_{ij}^2\, \omega^{[j]}$ mod dt on $U_i \cap U_j$

[around north pole: $\omega^{[0]} = \omega^+ - it\,\omega^3 + t^2 \omega^-$

$$\omega^\pm = -\frac{1}{2}\left(\omega^1 \mp i\omega^2\right)$$

* $\exists$ antiholomorphic involution $\sigma : Z \to Z$
covering $t \to -1/\bar{t}$

* $\forall x \in \mathcal{M}$, $\exists$ holomorphic section $\qquad \mathbb{P}^1 \xrightarrow{s_x} Z$
with normal bundle $\mathcal{O}(1)^{\oplus 2r}$

$s_x(\mathbb{P}^1)$ is the twistor line above x in opposite fibration

$$\mathbb{P}^1 \xrightarrow{s_x} Z \longrightarrow \mathcal{M}$$

Conversely, $(\mathcal{M}, ds^2)$ can be recovered from (Z, Ω):

e.g. around north pole:

$$\omega^{[0]} = \omega^+ - it\omega^3 + t^2\omega^-$$

read off Kähler form from ω^3.

Locally one can always choose complex Darboux coordinates

$\xi^\Lambda_{[i]}, \tilde{\xi}^{[i]}_\Lambda \qquad \Lambda = 1..r \qquad$ on $\mathcal{U}_i$

such that $\qquad \omega^{[i]} = d\xi^\Lambda_{[i]} \wedge d\tilde{\xi}^{[i]}_\Lambda$

The $\mathcal{O}(2)$-complex symplectic structure is encoded in complex symplectomorphisms on $\mathcal{U}_i \cap \mathcal{U}_j$

(analytic data, subject to reality conditions)

A useful way of parametrizing such cx symplectomorphisms is by a hol. funct. $\quad H^{[ij]}\left(\xi^\Lambda_{[i]}, \tilde{\xi}^{[j]}_\Lambda, t\right) \quad$ on $\mathcal{U}_i \cap \mathcal{U}_j$

such that

$$\begin{cases} \tilde{\xi}^{[i]}_\Lambda = \tilde{\xi}^{[j]}_\Lambda - \dfrac{\partial}{\partial \xi^\Lambda_{[i]}} H^{[ij]} \\[2ex] \xi^\Lambda_{[i]} = f^2_{ij}\, \xi^\Lambda_{[j]} + \dfrac{\partial}{\partial \tilde{\xi}^{[j]}_\Lambda} H^{[ij]} \end{cases}$$

$\{H^{[ij]}\}$ must satisfy consistency conditions on $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$,
reality conditions
and is ambiguous up to local symplecto on $\mathcal{U}_i$ and $\mathcal{U}_j$

(see eg APSV 0806.4620)

Finally, the moment map construction provides a useful source of holomorphic functions on Z:

<u>Thm</u> (Galicki, ...)

A tri-holomorphic isometry of $\mathcal{M}$ $\quad (\mathcal{L}_K \vec{\omega} = 0)$ lifts to a holomorphic section of $H^0(Z, \mathcal{O}(2))$

$$i_K \vec{\omega} = d\vec{\eta}$$

$$\eta^{[a]} = \eta^+ - it\,\eta^3 + t^2 \eta^- \quad \text{around } t=0$$

For toric HK manifolds (those admitting r commuting isometries) one can use the η's as coordinates ξ^Λ globally, so that $H^{[ij]}$ is independent of $\xi^\Lambda_{[i]}$

$\rightsquigarrow$ Legendre transform construction of toric HK metrics

Hitchin Karlhede Lindström Roček,

2.3 The twistor space of the rigid c-map metric

The c-map metric has a $U(1)^{2r}$ triholomorphic action

↳ we can get global Darboux coordinates on $\mathcal{Z}$

as moment maps : away from $t=0$, $t=\infty$

$$\Xi = \begin{pmatrix} \xi^\Lambda \\ \tilde{\xi}_\Lambda \end{pmatrix} = \underbrace{\begin{pmatrix} \zeta^\Lambda \\ \tilde{\zeta}_\Lambda \end{pmatrix}}_{=C} + \frac{iR}{2} \left[t \begin{pmatrix} \bar{X}^\Lambda \\ \bar{F}_\Lambda \end{pmatrix} - \frac{1}{t} \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} \right] \qquad (*)$$

Darboux coordinates around $t=0$, $t=\infty$

can be obtained by applying a complex symplectomorphism

generated by
$$H^{[10]} = R^2 F(t\xi^\Lambda)/t^2$$
$$H^{[1\infty]} = R^2 \bar{F}(\xi^\Lambda/t)/t^2$$

Eq $(*)$ shows that for a fixed complex structure, $\mathcal{Z}$ is the complexified torus $T_{\mathbb{C}} = \Gamma^* \otimes \underbrace{\mathbb{C}/2\pi\mathbb{Z}}_{=\mathbb{C}^\times}$

Define $\mathcal{X}_\gamma : \gamma \longmapsto \exp(2\pi i \langle \Xi, \gamma \rangle) \in \mathbb{C}^\times$
$$= \exp\left(2\pi i (q_\Lambda \xi^\Lambda - p^\Lambda \tilde{\xi}_\Lambda)\right)$$

The complex symplectic structure on $\mathcal{Z}$ is just

$$\omega = d\xi^\Lambda \wedge d\tilde{\xi}_\Lambda$$

Instanton corrections generically break $U(1)^{2r}$ to $\mathbb{Z}^{2r}$. If one restricts γ to lie in a Lagrangian sublattice of Γ, $U(1)^r$ is preserved, and the Legendre transform construction still applies. For brevity we skip this and show how the KS wall-crossing repairs the answer

2.4. The KS wall crossing formula

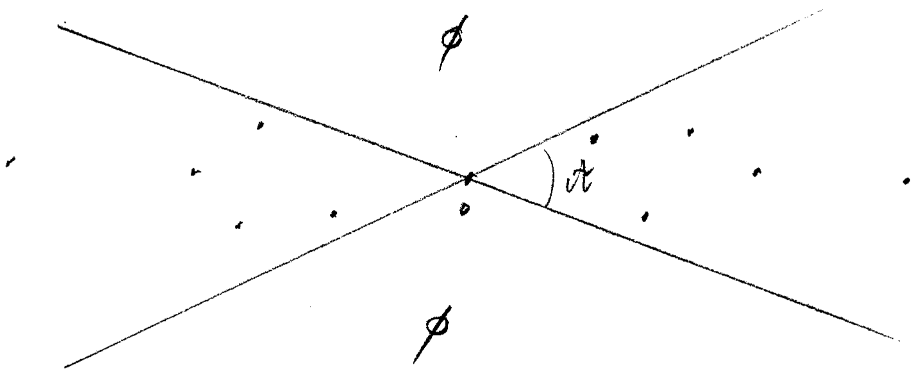
$\mathcal{B} \times \Gamma \xrightarrow{Z} \mathbb{C}$ central charge function

$\mathcal{B} \times \Gamma \xrightarrow{\Omega} \mathbb{Z}$ "BPS invariants"

$\Omega(\gamma; u)$ are locally constant, but may jump across codimension 1 walls in $\mathcal{B}$, where $\operatorname{Im} Z(\gamma_1)\bar{Z}(\gamma_2) = 0$, $\gamma \in \operatorname{span}(\gamma_1, \gamma_2)$

Assume that in an open subset $\mathcal{V} \subset \mathcal{B}$, (Ω, Z) satisfy the positive cone property:

$\exists$ angular sector $\mathcal{A} \subset \mathbb{C}$, width less than π, such that $\Omega(\gamma; u) = 0$ if $Z(\gamma) \notin \mathcal{A}$



let $\tilde{\Gamma} = \Gamma \cap Z^{-1}(\mathcal{A})$

Consider the pronilpotent algebra spanned by e_γ, $\gamma \in \tilde{\Gamma}$,

$$[e_\gamma, e_{\gamma'}] = (-1)^{\langle\gamma,\gamma'\rangle} \langle\gamma,\gamma'\rangle\, e_{\gamma+\gamma'}$$

Now define

$$U_{\mathcal{A}} \equiv \prod_{\gamma \in \tilde{\Gamma}} U_\gamma$$

product ordered such that $\arg Z_\gamma$ decreases from left to right

$$U_\gamma = \exp\left(\Omega(\gamma, u^a) \sum_{d=1}^{\infty} e_{d\gamma} / d^2 \right)$$

Claim: $U_{\mathcal{A}}$ is constant in the neighborhood $\mathcal{V}$.

Across a wall, the BPS rays $\ell_\gamma = \mathbb{R}^+ Z(\gamma)/|Z(\gamma)|$ will pass through each other and the order of factors U_γ will change, but $\Omega(\gamma, u^a)$ must change so as to leave $U_{\mathcal{A}}$ constant

Rk 1 – (outside the neighborhood $\mathcal{V}$, some states may leave the wedge $\mathcal{A}$, in which case $U_{\mathcal{A}} \mapsto U_\gamma U_{\mathcal{A}} U_\gamma^{-1}$.)

Rk 2 – It is convenient to collect factors $U_{k\gamma}$, $k>0$ into

$$V_\gamma = \prod_{k=1}^{\infty} U_{k\gamma} = \exp\left(\bar{\Omega}(\gamma, u^a)\, e_\gamma \right)$$

$$\bar{\Omega}(\gamma, u) = \sum_{m|\gamma} \frac{1}{m^2} \bar{\Omega}(\gamma, u)$$ are the "rational BPS invariants"

The wall crossing can then be reexpressed as

$$\bar{\Omega}^-(\gamma, u) = \sum_{n=1}^{\infty} \sum_{\substack{\gamma = \gamma_1 + \gamma_2 + \dots + \gamma_n \\ \gamma_i \in \tilde{P}}} g(\{\gamma_i\})\, \bar{\Omega}(\gamma_1, u) \dots \bar{\Omega}(\gamma_n, u)$$

The $g(\{\gamma_i\})$ are some universal combinatorial factors which have been computed (indep. from KS) by Joyce and Song, and which have been rederived from elementary physical considerations by Manschot BP Sen 1011.1258

2.5 The instanton corrected HK metric on $\mathcal{M}$

Reinterpret the KS formula in geometrical terms:

let $\sigma(\gamma)$ be a quadratic refinement of symplectic pairing:

$$\sigma(\gamma+\gamma') = \sigma(\gamma)\ \sigma(\gamma')\,(-1)^{\langle\gamma,\gamma'\rangle}$$

let $\quad \delta(\gamma) = \sigma(\gamma)\ e(\gamma)$; the algebra

$$[\,\delta(\gamma)\ ,\ \delta(\gamma')\,] = \langle\gamma,\gamma'\rangle\ \delta_{\gamma+\gamma'}$$

is recognized as the Lie algebra of Hamiltonian vector fields on a symplectic torus T of rank $2r$:

$$\delta_\gamma f = \{\, \mathcal{X}_\gamma, f \,\} \qquad \text{Poisson bracket}$$

$$\mathcal{X}_\gamma = \exp(\langle \Xi, \gamma\rangle\, 2\pi i)$$

The group elements $\quad U_\gamma = \exp\Big[\sum_{k=1}^{\infty} \sigma(k\gamma)\ \bar\Omega(k\gamma)\ \delta(k\gamma)\Big]$

are finite symplectomorphisms generated by

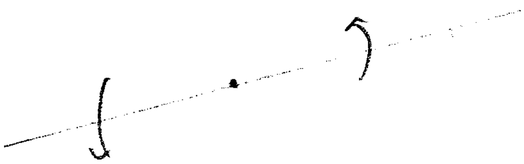
$$H^{[ij]}\big(\xi^\Lambda_{[i]}, \tilde\xi^{[j]}_\Lambda\big) = \sum_{k=1}^{\infty} \sigma(k\gamma)\ \bar\Omega(k\gamma)\ \exp\Big[2\pi i k\big(\xi^\Lambda_{[i]}\, q_\Lambda - \tilde\xi^{[j]}_\Lambda\, p^\Lambda\big)\Big]$$

across the BPS ray ℓ_γ in the t plane.

$$\| $$

$$\sum_{k=1}^{\infty} \sigma(k\gamma)\ \Omega(k\gamma)\ \mathrm{Li}_2\big(\mathcal{X}_{k\gamma}\big)$$

t



The instanton corrected Darboux coordinates are discontinuous across the BPS ray γ, according to

$$\mathcal{X}_{\gamma'} \longmapsto \mathcal{X}_{\gamma'} \left(1 - \sigma(\gamma) \mathcal{X}_\gamma \right)^{\Omega(\gamma) \langle \gamma, \gamma' \rangle}$$

Thus, they solve the system of integral equations

$$\mathcal{X}_\gamma = \mathcal{X}_\gamma^{sf} \exp\Big[-\frac{1}{2\pi i} \sum_{\gamma'} \Omega(\gamma', u) \langle \gamma, \gamma' \rangle \int_{\ell_{\gamma'}} \frac{dt'}{t'} \frac{t+t'}{t'-t} \log\left(1 - \sigma(\gamma') \mathcal{X}_{\gamma'}(t')\right) \Big]$$

where
$$\mathcal{X}_\gamma^{sf} = \exp\Big[2\pi i \left(q_\Lambda \zeta^\Lambda - p^\Lambda \tilde{\zeta}_\Lambda \right) + \frac{iR}{2} \left(t \bar{Z}_\gamma - \frac{1}{t} Z_\gamma \right) \Big]$$

To first subleading order in $\exp(-R)$, one may substitute $\mathcal{X}_{\gamma'} \to \mathcal{X}_{\gamma'}^{sf}$ on the rhs. The t' integral is governed by a saddle point:

$$\langle \mathcal{X}_\gamma^{sf} \rangle \sim \exp\Big[2\pi i \left(q_\Lambda \zeta^\Lambda - p^\Lambda \tilde{\zeta}_\Lambda \right) - 2\pi R |Z_\gamma| \Big]$$

as required. Moreover the metric is smooth across walls.

The singularities due to massless states in 4D seem also less severe, although a general analysis is lacking.

Lecture 3 – QK metrics on families of Abelian varieties

main ref APSV 0812.4219

1. Recap

In lecture 2, we constructed a 1-param family HK metric on a complex family of Abelian varieties, modelled on the Jacobian of a family of Riemann surfaces

$$\begin{array}{ccccl} t\in\mathbb{P}^1 & \longrightarrow & \mathcal{Z}_{4r+2} & \ni & (\xi^\Lambda,\tilde\xi_\Lambda,t) \\ & & \downarrow & & \\ C=(\xi^\Lambda,\tilde\xi_\Lambda)\in T_{2r} & \longrightarrow & \mathcal{M}_{4r} & & \\ & & \downarrow & & \\ & & \mathcal{B}_{2r}\ni u & & \end{array}$$

$T=\Gamma^*\otimes\mathbb{R}/2\pi\mathbb{Z}$

($\Gamma=H_1(\Sigma_u,\mathbb{Z})$
$\mathcal{B}$ = u-plane in Seiberg-Witten theory)

Period matrix $\tau_{\Lambda\Sigma}=\dfrac{\partial^2 F}{\partial X^\Lambda\partial X^\Sigma}$

$\mathrm{Im}\,\tau_{\Lambda\Sigma}>0$, $\tau_{\Lambda\Sigma}$ holomorphic on $\mathcal{B}$

The HK metric on $\mathcal{M}$ was encoded in the complex symplectic structure of the twistor space $\mathcal{Z}$, and depended on a set of BPS invariants

$\Omega(\gamma,u):\ \Gamma\times\mathcal{B}\backslash W\longmapsto\mathbb{Z}$

where W is the set of all "walls of marginal stability"

$W=\bigcup_{\gamma_1,\gamma_2\in\tilde\Gamma\times\tilde\Gamma}W(\gamma_1,\gamma_2)$ $\qquad\tilde\Gamma$: positive cone in Γ

$W_{\gamma_1,\gamma_2}=\{u\in\mathcal{B}\,/\ Z(\gamma_1)/Z(\gamma_2)\in\mathbb{R}^+\}$

satisfying the KS wall crossing condition:

$\prod_{\gamma\in\tilde\Gamma}U_\gamma$ [ordered such that $\arg Z_\gamma$ decreases from left to right] is constant across the wall

$$U_\gamma=\exp\Big[\Omega(\gamma,u)\sum_{d=1}^{\infty}e_{d\gamma}/d^2\Big]$$

$$[e_\gamma,e_{\gamma'}]=(-1)^{\langle\gamma,\gamma'\rangle}\langle\gamma,\gamma'\rangle\,e_{\gamma+\gamma'}$$

The key point was to notice that at fixed $t \in \mathbb{P}^1$,

$\mathcal{M}_t \simeq T_{\mathbb{C}} \equiv \Gamma^* \times \mathbb{C}^\times$ complexified torus

$\exists$ Darboux coordinates $\Xi = \xi^\Lambda, \tilde{\xi}_\Lambda \in \mathbb{C}/2\pi\mathbb{Z}$ such that

$\omega_+ = d\xi^\Lambda \wedge d\tilde{\xi}_\Lambda$ $\quad \Lambda = 1 .. r$

$\mathcal{X}_\gamma : \Gamma \times \mathcal{Z} \longrightarrow \mathbb{C}^\times$

$$\mathcal{X}_\gamma = \exp(2\pi i \langle \Xi, \gamma \rangle)$$
$$= \exp\left[2\pi i (q_\Lambda \xi^\Lambda - p^\Lambda \tilde{\xi}_\Lambda)\right] \in \mathbb{C}^\times$$

$$\mathcal{H} = \mathcal{H}_{sf} + \mathcal{H}_{inst}$$

- $\mathcal{H}_{sf} = C + \frac{iR}{2}\left(\frac{Z}{t} - \bar{Z} t\right)$ gives rise to the semiflat metric ; regular around equator, singularity at north and south pole can be cancelled by a symplecto generated by $H^{[1,0]} = F(t\xi^1)/t^2)$

- $\mathcal{H}_{inst}$ are exponentially suppressed corrections, discontinuous across BPS rays $\ell_\gamma = \mathbb{R}^+ Z(\gamma)$ in $\mathbb{P}_1$

Since $\delta(\gamma) \equiv \sigma(\gamma)\, e(\gamma)$ satisfies the Lie algebra of as Hamiltonian fields on $T_{\mathbb{C}}$

$$\left(\begin{array}{l} \sigma : \Gamma \mapsto \mathbb{Z}_2 \\ \sigma(\gamma+\gamma') = \sigma(\gamma)\,\sigma(\gamma')\,(-1)^{\langle \gamma, \gamma' \rangle} \\ \text{quadratic refinement} \end{array} \right)$$

$$\delta(\gamma) \cdot f = \{ \mathcal{X}_\gamma, f \}$$

at fixed $u \in \mathcal{B}$

It is natural to postulate that the discontinuity of $\mathcal{H}_{inst}$ across ℓ_γ is given by the finite symplectomorphism $\mathcal{U}_\gamma$

$$\mathcal{X}_{\gamma'} \longmapsto \mathcal{X}_{\gamma'} \left(1 - \sigma(\gamma)\, \mathcal{X}_\gamma\right)^{\Omega(\gamma,u) \langle \gamma, \gamma' \rangle}$$

The resulting ~~ex~~ symplectic structure on $\mathcal{Z}$ is then independent of u, thanks to wall crossing identity.

3.2 Families of Abelian varieties from CY_3 manifolds

[Physical context: hypermultiplet moduli space of type II string theory on a CY 3-fold X, or vector multiplet mod space of dual type II theory on $X \times S^1$]

3.2.1 Let $(X_u, \Omega_{3,0})$ a family of CY 3-folds with a choice of hol. 3-form, fibered over a base $\mathcal{B}$, $\dim_{\mathbb{C}} = h_{1,2}(X) + 1 = r$

Let $\Gamma = H_3(X_u, \mathbb{Z})$: symplectic lattice, rank r

$T_u = \Gamma^* \otimes \mathbb{R}/2\pi\mathbb{Z}$ is a family of real tori over $\mathcal{B}$

$Z(\gamma, u) = \oint_\gamma \Omega_{3,0} = q_\Lambda X^\Lambda - p^\Lambda F_\Lambda$ central charge function

The period matrix $\tau_{\Lambda\Sigma} = \partial_{X^\Lambda}\partial_{X^\Sigma} F$, corresponding to the Griffiths cx structure on T_u, varies holomorphically over $\mathcal{B}$, but has indefinite signature $(1, h_{1,2})$.

Instead, consider the Weil period matrix

$$\mathcal{N}_{\Lambda\Sigma} = \bar{\tau}_{\Lambda\Sigma} + 2i \frac{[\mathrm{Im}\,\tau \cdot X]_\Lambda \cdot [\mathrm{Im}\,\tau \cdot X]_\Sigma}{X \cdot \mathrm{Im}\,\tau \cdot X}$$

$$\mathrm{Im}\,\mathcal{N} < 0$$

Similarly, the Kähler metric on $\mathcal{B}$ with Kähler pot. $K = |\langle Z, \bar{Z}\rangle|^2$ has indefinite signature. Consider instead

$\mathcal{B}_0 = \mathcal{B}/\mathbb{C}^\times$ where $\mathbb{C}^\times$ acts by rescalings $\Omega_{3,0} \mapsto \Omega_{3,0}\, e^f$

$K = -\log |\langle Z, \bar{Z}\rangle|^2$ defines a (projective special) Kähler metric

Finally, consider a bundle of punctured disks $\mathcal{D}$ over $T \to \mathcal{M}_0 \to \mathcal{B}_0$ with metric $ds^2_{\mathcal{D}} = \frac{dR^2}{R^2} + \frac{1}{R^4}(D\sigma)^2$ where $dD\sigma = \omega_{T_u}$, the Kähler form on T_u

Thm 1 (Ferrara Sabharwal, "local c-map", see lec 1)

The $(4r+4)$-dim space $\mathcal{D} \to \mathcal{M}$ admits a (positive definite) QK metric

$$\begin{array}{ccc} \mathcal{D} & \to & \mathcal{M} \\ & & \downarrow \\ T & \to & \mathcal{M}_0 \\ & & \downarrow \\ & & \mathcal{B}_0 \end{array}$$

given locally by

$$ds^2_{\mathcal{M}} = ds^2_{\mathcal{D}} + ds^2_{\mathcal{B}_0} + \frac{1}{R^2} ds^2_{T_u}$$

$$\left(\text{recall } ds^2_{T_u} = -[\mathrm{Im}\,\mathcal{N}]^{\Lambda\Sigma}\left(d\tilde{\xi}_\Lambda - \mathcal{N}_{\Lambda\Lambda'} d\xi^{\Lambda'}\right)\left(d\tilde{\xi}_\Sigma - \bar{\mathcal{N}}_{\Sigma\Sigma'} d\xi^{\Sigma'}\right)\right)$$

↑ the inverse of $\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}$

This metric is accurate in the ∞ radius limit (or zero string coupling)

ex for a rigid CY manifold X, $\mathcal{B}_0 = \{pt\}$ and the above metric,

$$ds^2 = \frac{dR^2}{R^2} - \frac{|d\tilde{\xi} - \mathcal{N} d\xi|^2}{R^2\,\mathrm{Im}\,\mathcal{N}} + \frac{(d\sigma + \xi d\tilde{\xi} - \tilde{\xi} d\xi)^2}{R^4}$$

is the invariant metric on $U(2,1)/U(2)\times U(1)$

The "one-loop correction" induces a deformation to

$$d\tilde{s}_{\mathcal{M}} = 4\frac{R^2+2c}{R^2(R^2+c)} dR^2 + 4\frac{R^2+c}{R^2} ds^2_{\mathcal{B}_0} + \frac{ds^2_T}{R^2}$$

$$+ \frac{2c}{R^4}\left|Z(dC)\right|^2 + \frac{R^2+c}{16R^4(R^2+2c)} D\sigma^2$$

where
$$\begin{cases} d\left(\frac{D\sigma}{2}\right) = \omega_T + \frac{\chi(X)}{24}\omega_{\mathcal{B}_0} = \omega_T - 8\pi c\,\omega_{\mathcal{B}_0} \\ c = -\chi(X)/192\pi \end{cases}$$

↖ Kähler form on $\mathcal{B}_0$

↳ the punctured disk bundle is now non trivially fibered over $\mathcal{B}_0$

When $\chi(X)>0$, the metric is singular at $R^2 = -2c$

Similar to the HK story, one expects instanton corrections of order

$$\Omega(\gamma;u)\ \exp\left(-2\pi R\,|Z(\gamma,u)| + 2\pi i\,\langle C,\gamma\rangle\right)$$

controlled by BPS invariants $\Omega(\gamma,u):\ H_3(X,\mathbb{Z}) \mapsto \mathbb{Z}$, satisfying the KS wall-crossing formula.

In this context, $\Omega(\gamma,u)$ counts (with sign) the number of stable special Lagrangian submanifolds of X in homology class γ (more generally, stable objects in the Fukaya category of X)

In addition, one also expects $\mathcal{O}\left(\exp(-|H|R^2 + i\pi\sigma)\right)$ corrections from "Kaluza-Klein monopoles", the mathematical structure of which is still unclear.

3.2.2. There is a "mirror version" of the above construction, where $\mathcal{B}_0$ is now the moduli space of complexified Kähler structures on X parameterized by

$$u^a = b^a + i t^a = \int_{\gamma^a}(B + iJ)$$

γ^a basis of $H_2(X,\mathbb{Z})$

The lattice Γ is now

$$H_{\text{even}}(X,\mathbb{Z}) = \underset{1}{H_0} + \underset{\gamma^a}{H_2} + \underset{\gamma_a}{H_4} + \underset{\omega_X}{H_6}$$

with symplectic pairing

$$\langle\gamma,\gamma'\rangle = \int \omega \wedge \tau(\omega')$$

$\hookrightarrow$ 1 on

-1 on

The central charge function is governed by the prepotential

$$F(X) = -\kappa_{abc}\frac{X^aX^bX^c}{X^0} + \frac{c_{2a}}{24}X^aX^0 + \frac{1}{2}A_{ab}X^aX^b$$

$$+ \chi(X)\,\frac{\zeta(3)(X^0)^2}{2(2\pi i)^3} + F_{GW}$$

where F_{GW} depends on genus 0 Gromov-Witten invariants:

$$F_{GW} = -\frac{(X^0)^2}{(2\pi i)^3}\sum_{k_a\gamma^a \in H_2^+(X)} n^{(0)}_{k_a}\,\mathrm{Li}_3\left[e^{2\pi i k_a X^a/X^0}\right]$$

$\kappa_{abc} = \int \omega^a \wedge \omega^b \wedge \omega^c$; $A_{ab} \in \mathbb{Z}/2$ such that $\frac{1}{2}\kappa_{abc}p^bp^c - A_{ab}p^b \in \mathbb{Z}$ $\forall p^a \in \mathbb{Z}$

In the limit $R \to \infty$, the metric takes the same form as above, with the deformation parameter $c = + \chi(X)/192\pi$

The BPS invariants now count (with signs) the number of stable coherent sheaves E on X (more generally, stable objects in the derived category of X) with generalized character in homology class γ

$$Z(\gamma, u) = \int_{\hat{X}} e^{-(B+iJ)}\, ch(E) \sqrt{Td(X)}$$

3.2.3 Denoting $QK_c(X)$ the first construction, we have the $QK_k(X)$ the second,

- "Quantum mirror symmetry conjecture"

$$QK_c(X) = QK_k(\hat{X}) \quad \text{if } (X, \hat{X}) \text{ mirror pair}$$

- S-duality conjecture:

  $QK(X)$ admits an isometric action of $SL(2,\mathbb{Z})$ (or perhaps a finite index subgroup thereof) acting by $\tau \to \frac{a\tau+b}{c\tau+d}$ on $\tau = \zeta^0 + iR$, etc

  (Physically, this follows from S-duality of type IIB string theory)

3.3. Twistor techniques for QK manifolds

<u>Thm (Salamon, LeBrun)</u>:

if $(\mathcal{M}, ds^2)$ is QK_{4r}, $\exists$ non trivial fibrations

$$\mathcal{M} \to \mathcal{Z} \xrightarrow{p} \mathbb{P}_1 \,, \quad \mathcal{M} \to \mathcal{Z} \xrightarrow{s} \mathbb{P}_1$$

such that $\mathcal{Z}$ carries a canonical complex structure

+ complex contact structure

+ antiholomorphic involution covering $t \to -1/\bar{t}$

and such that the complex contact structure is given by the kernel of the $[\mathcal{O}(2)$-twisted$]$ $(1,0)$ form on $\mathcal{Z}$

$$Dt \equiv dt + p_+ - i p_3 t + p_- t^2$$

$\vec{p}$: $SU(2)$ part of Levi-Civita connecti[on]

and the QK metric on $\mathcal{M}$ can be recovered from this data.

Locally, $\exists$ "contact potential" $\Phi : \mathcal{Z} \to \mathbb{C}$, $\partial_{\bar{t}} \Phi = 0$,

such that

$$\mathcal{X} = 2\, e^{\Phi} \frac{Dt}{t} \quad \text{is a holomorphic 1-form}$$

. $\exists$ "Darboux coordinates" $(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)$ $\Lambda = 1 .. r-1$

such that

$$\mathcal{X} = d\alpha + \xi^\Lambda d\tilde{\xi}_\Lambda - \tilde{\xi}_\Lambda d\xi^\Lambda$$

The complex contact structure on $\mathcal{Z}$ can be specified by complex contact transformations on the overlaps of local Darboux coordinate patches:

$$\mathcal{X}^{[i]} = f_{ij}^2 \, \mathcal{X}^{[j]}$$

<u>Rk</u> . $K = \mathrm{Re}\,\Phi + \log \frac{1+t\bar{t}}{|t|}$ provides a Kähler potential for the canonical Kähler-Einstein metric

$$\frac{ds^2}{2} = \frac{|Dt|^2}{(1+t\bar{t})^2} + \underbrace{\frac{\nu}{2}}_{\text{constant}} ds^2_{\mathcal{M}}$$

By solving the gluing conditions

$$\mathcal{X}^{[i]} = f_{ij}^{2}\, \mathcal{X}^{[j]}$$

for the $\begin{pmatrix} \text{Darboux coordinates} \\ \text{Contact potential} \end{pmatrix}$ as Taylor series on $\mathbb{P}_1$, one can recover the QK metric

* Just as in HK case, the moment map construction provides a source of local holomorphic functions: if K is a Killing field on $\mathcal{M}$,

$$\eta = e^{\phi}\left(\eta_+/t - i\eta_3 + \eta_- t \right)$$

gives a global section of $H^0(Z, \mathcal{O}(2))$

Here $\vec{\eta} = \frac{1}{2}\left(\vec{r} + K \lrcorner \vec{p}\right)$ where $\vec{r}$ is such that $\mathcal{L}_K \vec{\omega} + \vec{r} \wedge \vec{\omega} = 0$

Moreover, K lifts to a holomorphic vector field K_Z on Z, such that

$$K_Z \lrcorner \mathcal{X} = \eta$$

* In the presence of one Killing field on $\mathcal{M}$, one can choose Darboux coordinates such that the Reeb vector is $\partial/\partial\alpha$ globally. The contact transformations then reduce to symplectomorphisms of $(\xi^\Lambda, \tilde{\xi}_\Lambda)$

* QK_{4r} manifolds with $r+1$ commuting isometries are amenable to the Legendre transform method. (their Swann bundle is a toric HK_{4r+4} manifold)

* Infinitesimal perturbations of QK manifolds are classified by $H^1(Z, \mathcal{O}(2))$

3.4 Twistorial description of the "local c-map" metric

* The local c-map metric admits an isometric action of

$$\mathbb{R}^+ \times \left(S_1 \ltimes T^{2r} \right)$$

$T^{2r} \to$
$$\left.\begin{array}{l} \partial/\partial\zeta^\Lambda - \tilde{\zeta}_\Lambda \dfrac{\partial}{\partial\sigma} \\ \partial/\partial\tilde{\zeta}_\Lambda + \zeta^\Lambda \dfrac{\partial}{\partial\sigma} \end{array}\right\}$$
generate a Heisenberg algebra

$S_1 \to \partial/\partial\sigma$

$\mathbb{R}^+ \to R\partial_R + \zeta\partial_\zeta + \tilde{\zeta}\partial_{\tilde{\zeta}} + 2\sigma\partial_\sigma$: Euler field

The moment maps for T^{2r} lead to $2r$ complex Darboux coordinates

$$\boxed{H}^{sf} = \begin{pmatrix} \xi^\Lambda \\ \tilde{\xi}_\Lambda \end{pmatrix} = C + \frac{iR}{2}\left(\frac{Z}{t} - \bar{Z}\, t \right)$$

The moment map for the Euler field leads to

$$\alpha^{sf} = \sigma + \frac{iR}{2}\left(\frac{\langle Z, C\rangle}{t} - \langle \bar{Z}, C\rangle \right)$$

Such that the contact 1-form is $\mathcal{X} \propto d\alpha + \xi^\Lambda d\tilde{\xi}_\Lambda - \tilde{\xi}_\Lambda d\xi^\Lambda$

* The "one-loop deformed local c-map metric" is described by the same Darboux coordinates, except for an additional log in α:

$$\alpha^{sf,1\text{-loop}} = \sigma + \frac{iR}{2}\left(\frac{\langle Z, C\rangle}{t} - \langle \bar{Z}, C\rangle\, t \right) - \frac{i\chi}{24\pi}\log t$$

The singularity at $t=0$, $t=\infty$ can be removed by a contact transformation

Rk. α is ambiguous mod $\frac{\chi}{12}$, whereas $SL(2,\mathbb{Z})$ requires that $\alpha \in \mathbb{C}/2\mathbb{Z}$. This is related to the $\frac{\chi}{24}\,\omega_{\mathcal{B}_0}$ term in the curvature of the $U(1)_\sigma$ circle bundle. There should be a way to fix this ambiguity

. Ignoring this subtlety, $Z = \mathbb{C}^\times \ltimes T_{\mathbb{C}}$, "complexified twisted torus"

3.5 D-instanton corrections

In the absence of "Kaluza Klein monopoles", the continuous isometry ∂_σ stays unbroken. The contact transformations reduce to symplectomorphisms of the twisted torus T_C, supplemented by suitable shift of α.

Given a set of BPS invariants $\Omega(\gamma,u)$, it is natural to use the same symplectomorphisms U_γ as in the HK story: across a "BPS ray" $\ell_\gamma = \mathbb{R}^+ Z(\gamma) \subset \mathbb{P}_1$,

$$\mathcal{X}_{\gamma'} \mapsto \mathcal{X}_{\gamma'} \left(1 - \sigma(\gamma)\, \mathcal{X}_\gamma\right)^{\langle\gamma,\gamma'\rangle\, \Omega(\gamma,u)}$$

$$\alpha \mapsto \alpha - \frac{\sigma(\gamma)\,\Omega(\gamma)}{2\pi^2}\, L(\mathcal{X}_\gamma)$$

where $\mathcal{X}_\gamma = \exp\left(2\pi i \langle \Xi, \gamma\rangle\right)$

$L(z) = \mathrm{Li}_2(z) + \frac{1}{2}\log z \log(1-z)$ is Rogers dilog.

The identity $L(x) + L(y) = L(xy) + L\left(\frac{x(1-y)}{1-xy}\right) + L\left(\frac{y(1-x)}{1-xy}\right)$ and more generally, the classical limit $\hbar \to 0$ of the motivic version of the KS formula guarantees that the complex structure thus defined is independent of the choice of u.

One way to understand this isomorphism between QK and HK constructions is that the Killing vector ∂_σ lifts to a triholomorphic isometric action on the Swann bundle.
The HK quotient with respect to this action (at arbitrary non zero level) leads to a HK manifold $\tilde{\mathcal{M}}$, equipped with a canonical hyperholomorphic line bundle $\mathcal{E}$

$$\begin{tikzcd}
 & \mathcal{S} \arrow[dl] \arrow[d, "\eta_+=0"] \arrow[ddr, "///\partial/\partial r"'] & \\
 & \mathcal{E} \arrow[dr, "\partial/\partial\alpha"] & \\
\mathcal{M}_{4r} & & \tilde{\mathcal{M}}_{4r}
\end{tikzcd}$$

The HK manifold $\tilde{\mathcal{M}}$ has signature $(4, 4r-4)$, and is exactly the one that would follow from the GMN construction.

As in the HK story, the QK metric on $\mathcal{M}$ can be constructed by solving a system of integral equations

$$\mathcal{X}_\gamma = \mathcal{X}_\gamma^{sf} \exp\left[-\frac{1}{2\pi i} \sum_{\gamma'} \Omega(\gamma,\gamma') \langle \gamma,\gamma'\rangle \int_{\ell_{\gamma'}} \frac{dt'}{t'} \frac{t+t'}{t'-t} \log\left(1-\sigma(\gamma')\mathcal{X}_{\gamma'}(t')\right)\right)$$

$$\begin{cases} e^{\phi} = \left(\frac{R^2}{2} K + \frac{\chi}{96\pi}\right) - \frac{iR}{16\pi^2} \sum_\gamma \Omega(\gamma)\,\sigma(\gamma) \int \frac{dt}{t} \left(\frac{Z(\gamma)}{t} - \bar{Z}(\gamma) t\right) \log\left(1-\sigma(\gamma)\mathcal{X}_\gamma\right) \\ \alpha = \alpha^{sf} + \sum_\gamma \Omega(\gamma)\,(\dots) + \sum_{\gamma,\gamma'} \Omega(\gamma)\Omega(\gamma')\langle\gamma,\gamma'\rangle\,(\dots) \end{cases}$$

This can be solved iteratively, however the resulting series is divergent, due to exponential growth of Ω:

$\Omega(\gamma) \underset{|\gamma|\to\infty}{\sim} \exp|\gamma|^2$ hence $\sum_\gamma \Omega(\gamma) \exp(-R|\gamma|)$ has ambiguity $\mathcal{O}(e^{-R^2})$

3.6 Beyond D-instantons

- $\mathcal{O}(e^{-R^2})$ effects are needed for S-duality:

  $\begin{pmatrix} \tilde{\xi}_0 \\ \alpha/2 \end{pmatrix}$ transform as a doublet under $SL(2)$ (plus stuff)

  so $\mathcal{X}_\gamma = \exp\left(2\pi i \left(p_0 \tilde{\xi}_0 + \cdots\right)\right)$ is mapped under $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

  into $\exp(-i\pi k\alpha + \cdots)$: this breaks the continuous translations along σ to a discrete group, $\sigma \in \mathbb{R}/2\mathbb{Z}$

- In the sector with $p^0 = k = 0$ (ie dividing out by translations with respect to $\tilde{\xi}_0$, σ), the above construction should already preserve an $SL(2,\mathbb{Z})$ isometric action.

  This is clear in the sector with $p^0 = k = p^a = 0$, since S-duality maps the GW contributions to F into instanton contributions of coherent sheaves supported on curves. Uncovering S-duality in sector with $p^a \neq 0$ will presumably involve theta series for indefinite lattices $\hookrightarrow$ Mock theta series.

- The KKMonopole contributions ($k \neq 0$) are also expected to involve theta series on T, or indefinite theta series on $T_{\mathbb{C}}$. Perhaps the KS/CV [Cecotti-Vafa] wall-crossing formula under development by GMN will give us a hint on the proper mathematical structure.

THE END