REPLICAS FOR RFIM

We will divide the sum by two to count each spin pair only once

$$H_h = -\frac{J}{2N}\sum_{i\neq j} s_i s_j - \sum_i h_i s_i$$

we called the field variance \sigma

$$P(h_i) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{h_i^2}{2\sigma^2}}$$

0 – RESCALE $h_i \to \sigma_h \cdot h_i \Rightarrow P(h_i) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{h_i^2}{2}}$

1 – STARTING FROM THE LATTICE VERSION WHERE $\sum_{\langle ij\rangle}$ AND COUPLING STRENGTH IS $O(1) \Rightarrow$ WE RESCALE $\to \frac{J}{N}$ & TAKE $\sum_{i\neq j}$ TO DO THE MEAN-FIELD CASE.

2 – REPLICAS

$$[-\beta f]_h = \frac{1}{N}[\ln Z_h]_h = \frac{1}{N}\lim_{n\to 0}\left[\frac{Z_h^n - 1}{n}\right]_h$$

$$-\beta N [f]_h = \lim_{n\to 0}\frac{[Z_h^n]_h - 1}{nN}$$

$$[Z_h^n] = \left[\sum_{\{s_i^a\}} e^{-\beta H_h[\{s_i^a\}]}\right]_h$$

CALL THEM η_i

$$= \sum_{\{s_i^a\}} e^{\frac{\beta J}{2N}\sum_{i\neq j}\sum_{a=1}^{n} s_i^a s_j^a}\left[e^{-\beta\sigma_h \sum_i h_i \sum_{a=1}^{n} s_i^a}\right]_h$$

NOTE: SUM OVER REPLICA INDICES IN EXPONENTIAL

$$\left[e^{-\beta \sigma_h \sum_i z_i \sum_{a=1}^{n} s_i^a} \right]_h = \prod_i \int \frac{dz_i}{\sqrt{2\pi}}\, e^{-\frac{z_i^2}{2} - \beta \sigma_h \sum_i z_i \sum_{a=1}^{n} s_i^a}$$

$$= \prod_i e^{\frac{(\beta \sigma_h)^2}{2} (s_i^a)^2}$$

$$[Z_h^n] = \sum_{\{s_i^a\}} e^{\frac{\beta J}{2N} \sum_{a=1}^{n} \underbrace{\sum_{i \neq j} s_i^a s_j^a}_{\sum_{ij} s_i^a s_j^a - \sum_i \underbrace{(s_i^a)^2}_{=1}} + \frac{(\beta \sigma_h)^2}{2} \sum_i \left(\sum_{a=1}^{n} s_i^a \right)^2}$$

NOTE: THE SUM IS WITHIN THE SQUARE $\Rightarrow$ REPLICAS INTERACT.

$$= \sum_{\{s_i^a\}} e^{\frac{\beta J}{N} \sum_{a=1}^{n} \left(\sum_i s_i^a \right)^2 + \left[\frac{(\beta \sigma)^2}{2} - \cancel{\frac{\beta J}{N}} \right] \sum_i \left(\sum_{a=1}^{n} s_i^a \right)^2}$$

~~ALTHOUGH BOTH PROP TO SAME~~ ~~$\sum_i (s_i^a)^2$~~ ~~FACTOR, THE $1/N$ MAKES 2nd NEGLIGIBLE~~

$$\xrightarrow[N \gg 1]{} \sum_{\{s_i^a\}} e^{\frac{\beta J}{N} \sum_{a=1}^{n} \left(\sum_i s_i^a \right)^2 + \frac{(\beta \sigma_h)^2}{2} \sum_i \left(\sum_{a=1}^{n} s_i^a \right)^2}$$

- NOW, USE HUBBARD-STRATONOVICH OR GAUSSIAN DECOUPLING

$$e^{\lambda b^2} = \int \frac{dz}{\sqrt{2\pi}}\, e^{\underbrace{-\frac{1}{2} z^2 + \sqrt{2\lambda}\, bz - \lambda b^2}_{-\frac{1}{2}\left(z - \sqrt{2\lambda}\, b\right)^2} + \lambda b^2}$$

TO DECOUPLE THE 1st TERM IN THE EXPONENTIAL (i.e., FROM A DOUBLE SUM $\sum_{ij}$ TO A SINGLE ONE.

$$e^{\frac{\beta J}{N}\left(\sum_i s_i^a\right)^2} = \int_{-\infty}^{\infty} \frac{dz_a}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z_a^2 + \sqrt{\frac{2\beta J}{2N}}\left(\sum_i s_i^a\right) z_a}$$

$\frac{\beta J}{N} \to \lambda$, $\left(\sum_i s_i^a\right)^2 \to b^2$

FOR EACH a INDEX $\Rightarrow$ NEED AN INDEX a IN z

AND $[Z^n]$ BECOMES

$$[Z_h^n] = \sum_{\{s_i^a\}} \int \frac{\prod_a dz_a}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z_a^2 + \sqrt{\frac{2\beta J}{2N}}\left(\sum_i s_i^a\right) z_a} \times e^{\frac{(\beta W_h)^2}{2}\sum_i\left(\sum_{a=1}^{n} s_i^a\right)^2}$$

FOR CONVENIENCE, I FACTORIZED THE EXPONENTIALS

$$= \int_{-\infty}^{\infty} \prod_{a=1}^{n} \frac{dz_a}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\sum_{a=1}^{n} z_a^2} \sum_{\{s_i^a\}} e^{\sqrt{\frac{2\beta J}{2N}}\sum_{a=1}^{n} s_i^a z_a + \frac{(\beta W_h)^2}{2}\sum_{a=1}^{n} s_i^a \sum_{b=1}^{n} s_i^b}$$

IT'S A MULTIPLE INTEGRAL, THERE ARE n OF THEM

SINCE THE i-SITES ARE DECOUPLED $\Rightarrow$ IT'S A MUTE INDEX AND ONE CAN ELIMINATE IT, AND CONSIDER ONLY ONE SUM $\sum_{s^a}$ TO THE POWER N:

$$\left(\sum_{\{s^a\}} e^{\sqrt{\frac{2\beta J}{2N}}\sum_{a=1}^{n} s^a z_a + \frac{(\beta W_h)^2}{2}\sum_{a=1}^{n} s^a \sum_{b=1}^{n} s^b}\right)^N$$

AND ONE WRITES $[Z^n]$ IN A MORE COMPACT FORM:

$$[Z_h^n] = \left(\frac{1}{2\pi}\right)^{n/2} \int \prod_{a=1}^{n} dz_a\, e^{-\frac{1}{2}\sum_a z_a^2 + N\ln Z_1(\{z_a\})}$$

WITH $$\mathcal{Z}_1(\{z_a\}) = \sum_{\{s^a\}} e^{\sqrt{\frac{2\beta J}{2N}} \sum_{a=1}^{n} s^a z_a + \frac{(\beta h)^2}{2} \sum_{ab=1}^{n} s^a s^b}$$

WE SEEM TO HAVE USELESSLY TRADED THE INT BTW $(ij) \to$ INT BTW (ab) AND NOW WE HAVE ANOTHER PART SUM TO COMPUTE. LET'S GO AHEAD.

henceforth h is \sigma_h but I cannot write on the pages anymore.

- RESCALE $z_a \longrightarrow \sqrt{N}\, z_a$

$$[Z_h^n] = \left(\frac{N}{2\pi}\right)^{n/2} \int \prod_{a=1}^{n} dz_a \; e^{-N\left[\frac{1}{2}\sum_a z_a^2 - \ln \mathcal{Z}_1(\{z_a\})\right]}$$

$$\mathcal{Z}_1(\{z_a\}) = \sum_{\{s^a\}} e^{\sqrt{\frac{2\beta J}{2}} \sum_{a=1}^{n} s^a z_a + \frac{(\beta h)^2}{2} \sum_{ab=1}^{n} s^a s^b}$$

- NOW, $\overset{\rightarrow\, N \to \infty \text{ AND}}{}$ ONE CAN EVALUATE THE INT BY STEEPEST DESCENT (EVEN THOUGH THERE ARE n VARIABLES OVER WHICH OPTIMISE). ASSUME THAT THE EXTREMA WE ARE LOOKING FOR ARE SUCH THAT

$$z_a^{EXT} = z^{EXT} \quad \text{INDEP. OF } a \quad \Longrightarrow$$

$$z_a^{EXT} = \left. \frac{\frac{\partial \mathcal{Z}_1(\{z_a\})}{\partial z_a}}{\mathcal{Z}_1(\{z_a\})} \right|_{z_a^{EXT}}$$

SOLUTION INDEP OF a

$$= \frac{\sum_{\{s^a\}} \left(\sqrt{2\beta J}\, s^a\right) e^{\sqrt{\frac{2\beta J}{2}} \sum_a s^a z_a + \frac{(\beta h)^2}{2} \sum_{ab} s^a s^b}}{\mathcal{Z}_1(\{z_a\})}$$

WE SEE THAT $\quad z_a^{EXT} = \sqrt{\frac{2\beta J}{2}} \; \langle s^a \rangle_{Z_1(\{z_a^{EXT}\})}$

IF, MOREOVER, $\quad z_a^{EXT} = z^{EXT} \implies$

$$z^{EXT} = \sqrt{\frac{2\beta J}{2}} \; \langle s^a \rangle_{Z_1(\{z^{EXT}\})} \qquad \text{INDEP OF } a.$$

REPLACE NOW z_a BY z^{EXT} IN $[Z_h^n]$ AND, FURTHERMORE, IN $[f_R]$:

$$-\beta\,[f_R] = \lim_{\substack{n\to 0\\ N\to\infty}} \frac{1}{nN}\left\{ \overbrace{\left(\frac{N}{2\pi}\right)^{n/2}}^{\text{GOES TO } 1 \text{ FOR } n\to 0} e^{-N\left[\frac{1}{2}n\left(z^{EXT}\right)^2 - \ln Z_1\left(z^{EXT}\right)\right]} - 1 \right\}$$

$$= \lim_{\substack{n\to 0\\ N\to\infty}} \frac{1}{nN}\left\{ e^{-N\left[\frac{1}{2}n\left(z^{EXT}\right)^2 - \ln Z_1\left(z^{EXT}\right)\right]} - 1 \right\}$$

ONE CAN DECOUPLE NOW THE REPLICA INDICES IN $Z_1(z^{EXT})$ TO COMPUTE THE SUM $\sum_{\{s^a\}}$ EXPLICITLY.

$$Z_1\left(z^{EXT}\right) = \sum_{\{s^a\}} e^{\sqrt{\frac{2\beta J}{2}}\, z^{EXT} \sum_a s^a + \frac{(\beta h)^2}{2} \sum_{ab=1}^{n} s^a s^b}$$

$$= \sum_{\{s^a\}} e^{\sqrt{\frac{2\beta J}{2}}\, z^{EXT} \sum_a s^a} \int \frac{d\omega}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\omega^2} \prod_a e^{\omega \beta h\, s^a}$$

NOW WE HAVE DECOUPLED THE $\{s^a\}_s$ AND WE CAN COMPUTE THE $\sum_{\{s^a\}}$:

$$\sum_{\{s^a\}} e^{\sqrt{2\beta J}\, z^{EXT} s^a + \omega \beta h s^a}$$

FOR EACH $a \Rightarrow n$ IDENTICAL PART. SUMS, LEADING TO

$$\Rightarrow \left[2\,\mathrm{ch}\left(\sqrt{2\beta J}\, z^{EXT} + \omega\beta h \right) \right]^n \Rightarrow$$

$$Z_n(z^{EXT}) = \int \frac{d\omega}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\omega^2} \underbrace{\left[2\,\mathrm{ch}\left(\sqrt{2\beta J}\, z^{EXT} + \omega\beta h \right) \right]^n}_{e^{n \ln\left(2\,\mathrm{ch}\left(\sqrt{2\beta J}\, z^{EXT} + \omega\beta h\right)\right)}}$$

AND, FINALLY,

$$-\beta\,[f_{eq}] = \lim_{\substack{n \to 0 \\ N \to \infty}} \frac{1}{nN} \left\{ e^{-N\left[\frac{1}{2}n\left(z^{EXT}\right)^2 - \ln \int \frac{d\omega}{\sqrt{2\pi}} e^{-\frac{1}{2}\omega^2} \underbrace{e^{n \ln\left(2\,\mathrm{ch}\left(\sqrt{2\beta J}\, z^{EXT} + \omega\beta h\right)\right)}}_{A}\right]} - 1 \right\}$$

AND WE WILL USE $\langle\langle \cdot \rangle\rangle$ TO INDICATE THE AVERAGE OVER THE GAUSSIAN WEIGHT OF THE NEW AUXILIARY VARIABLE ω.

FIRST WE TAKE $n \to 0$ IN THE EXPONENTIAL:

$$A \equiv e^{n \ln\left(2\,\mathrm{ch}\left(\sqrt{2\beta J}\, z^{EXT} + \omega\beta h\right)\right)} \simeq 1 + n \ln\left(2\,\mathrm{ch}\left(\sqrt{2\beta J}\, z^{EXT} + \omega\beta h\right)\right)$$

$\rightarrow$ THE 1 GOES BELOW $\langle\langle \cdot \rangle\rangle$ $\nearrow$ $\langle\langle 1 \rangle\rangle = 1$

→ THE SECOND TERM REMAINS AND WE HAVE

$$\ln \left\langle\!\left\langle \left[\overbrace{1 + n \ln \left(2 \,\mathrm{ch} \left(\sqrt{2\beta J}\; z^{EST} + \beta h w \right) \right)}^{A} \right] \right\rangle\!\right\rangle$$

$$\xrightarrow[n \to 0]{} n \left\langle\!\left\langle \ln \left(2 \,\mathrm{ch} \left(\sqrt{2\beta J}\; z^{EXT} + \beta h w \right) \right) \right\rangle\!\right\rangle$$

WHERE WE USED $\ln(1+\epsilon) \simeq \epsilon$

SO, WITH THE $n \to 0$ WE GOT RID ON ONE $\ln$ FUNCTION

NOW, THE FULL EXPONENT IS PROP TO n AND WE TAKE AGAIN $n \to 0$:

$$e^{-Nn \text{ Exponent}} \simeq 1 - Nn \text{ Exponent}$$

THE 1 CANCELS THE (-1) AND THE Nn SIMPLIFY WITH THE $\frac{1}{n}$:

$$-\beta \left[f_h \right] = \underbrace{\lim_{\substack{n \to 0 \\ N \to \infty}}}_{} - \left\{ + \frac{1}{2} \left(z^{EXT} \right)^2 - \left\langle\!\left\langle \ln \left(2 \,\mathrm{ch} \left(\sqrt{2\beta J}\; z^{EXT} + \beta h w \right) \right) \right\rangle\!\right\rangle \right\}$$

THERE'S NO MORE DEPENDENCE ON (n, N)
$\Rightarrow$ IRRELEVANT

$$\beta \left[f_h \right] = \frac{1}{2} \left(z^{EXT} \right)^2 - \left\langle\!\left\langle \ln \left(2 \,\mathrm{ch} \left(\sqrt{2\beta J}\; z^{EST} + \beta h w \right) \right) \right\rangle\!\right\rangle$$

RECALLING THAT $z^{EXT} = \sqrt{2\beta J}\, m$, WE REPLACE AND

$$[f_J] = Jm^2 - \frac{1}{\beta}\langle\langle \ln\left(2\,\mathrm{ch}\left(\tfrac{2}{2}\beta J m + \beta h w\right)\right)\rangle\rangle$$ AVERAGED FREE-ENERGY DENSITY

cfr W/ ISING MODEL IN A FIELD.

$$\frac{\partial [f_J]}{\partial m} = 0 \Rightarrow 0 = \cancel{2}Jm - \frac{1}{\cancel{\beta}}\langle\langle \frac{\cancel{2}\,\mathrm{sh}(2\beta J m + \beta h w)}{\cancel{2}\,\mathrm{ch}(2\beta J m + \beta h w)}\,\cancel{2\beta J}\rangle\rangle$$

Everywhere the 1/2 comes from the normalisation of the sum over neighbours in the Hamiltonian, that is more usually written with the red 1/2. Then the critical temperature is changed as well, since it's like making J->J/2

$$m_{EXT} = \langle\langle \mathrm{th}\left(\tfrac{2}{2}\beta J m_{EXT} + \beta h w\right)\rangle\rangle$$ SADDLE-POINT EQ.

- $m_{EXT} = 0$ IS A SOLUTION $\forall (\beta J, \beta h)$

  INDEED $0 = \langle\langle \mathrm{th}(\beta h w)\rangle\rangle$ AND THE r.h.s $= 0$ SINCE $P(w)$ IS SYMM WRT 0 WHILE $\mathrm{th}\,\beta h w$ IS NOT.

  $\boxed{m_{PM} = 0}$ PARAM SOLUTION.

- ASSUME A 2nd ORDER PHASE TRANSITION, ie, WITH $m_{FM} \simeq 0$ CLOSE TO T_c & EXPAND THE th

$$m_{EXT} \approx \langle\langle \, th(\beta h w) + \frac{ch^2 \beta h w - sh^2 \beta h w}{ch^2 \, \beta h w} \, 2\beta J \, m_{EXT} \rangle\rangle$$

$$m_{EXT} = \langle\langle \, \underbrace{\cancel{th(\beta h w)}}_{\to 0} \, \rangle\rangle + 2\beta J \, m_{EXT} \, \langle\langle \frac{1}{ch^2 \beta h w} \rangle\rangle$$

2

$$\Rightarrow \quad \boxed{1 = 2\beta_c J \, \langle\langle \frac{1}{ch^2 \beta_c h w} \rangle\rangle}$$

2

IT'S A CONDITION ON β_c GIVEN J AND σ_h THAT IS IMPLICIT IN THE $p(h) \to p(w)$ WITH UNIT VARIANCE BUT h FACTOR IS $h = \sigma_h$ OUTSIDE THAT WE HAVE TO USE

PARAMETERS $(\beta_c J)$, $(\beta_c h)$

IN THIS WRITING

OR $x_c = \beta_c J$ $\quad y_c = \beta_c h$ $\quad$ OR $\quad y_c = \frac{\beta_c}{J} J h$

$$= (\beta_c J)\left(\frac{h}{J}\right)$$

$$= x_c \left(\frac{h}{J}\right)$$

With the factors of 2 cancelled it's a nicer expression

$$\boxed{1 = 2\beta_c J \, \langle\langle \frac{1}{ch^2 (\beta_c J)\left(\frac{h_c}{J}\right)} \rangle\rangle}$$

2