Out of Equilibrium Dynamics of Complex Systems

Leticia F. Cugliandolo

Sorbonne Universités, Université Pierre et Marie Curie

Laboratoire de Physique Théorique et Hautes Energies

Institut Universitaire de France

leticia@lpthe.jussieu.fr

www.lpthe.jussieu.fr/˜leticia/seminars

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Plan of Lectures

1. Introduction
2. **Coarsening processes**
3. Formalism
4. Dynamics of disordered spin models

References

— Phase ordering kinetics & critical dynamics

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F. Corberi & A. Politi eds., *Coarsening dynamics*, Comptes Rendus Physique **16** (2015).

P. Krapivsky, S. Redner and E. Ben-Naim, *A kinetic view of statistical physics* (Cambridge University Press, 2010).

M. Henkel and M. Pleimling, *Non-Equilibrium Phase Transitions : Volume 2 : Ageing and Dynamical Scaling Far from Equilibrium*, (Springer, 2010).

L. F. Cugliandolo, *Dynamics of glassy systems*, Les Houches Session 77, arXiv :cond-mat/0210312.

U. C. Tauber, *Critical Dynamics : A Field Theory Approach to Equilibrium and Non-Equilibrium Scaling Behavior* (Cambridge University Press, 2014)

Plan of the lecture

1. The phenomenon
2. Theoretical setting
3. Critical and sub-critical quenches
4. Dynamic scaling
5. Dynamic universality classes
6. Two-time correlations and ageing
7. Two-time responses and loss of memory
8. Mean-field models
9. Modern studies

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Phenomenon

The talk focuses on a very well-known example

Dynamics following a change of a control parameter

- If there is an equilibrium phase transition, the **equilibrium phases** are known on both sides of the transition.

  i.e. the asymptotic state is known.
- For a purely dynamic problem, the **absorbing states** are known.
- The **dynamic mechanism** towards equilibrium (or the absorbing states) is understood the systems try to order locally in one of the few competing states.

Interests and goals

Practical interest, *e.g.*

- Mesoscopic structure effects on the opto-mechanical properties of phase separating glasses
- Cooling rate effects on the density of topological defects in cosmology and condensed matter

Fundamental interest, *e.g.*

- A theoretical problem beyond perturbation theory.
- Are there growth phenomena in problems with yet unknown dynamic mechanisms ? **e.g. glasses**
- Generic features of macroscopic systems out of equilibrium ?

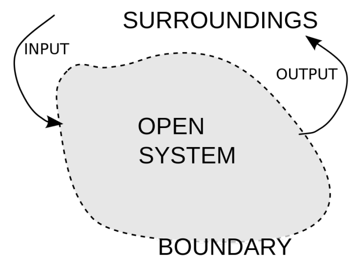
Context

Open systems

Our interest is to describe the dynamics of a classical (or quantum) system coupled to a classical (or quantum) environment.

The Hamiltonian of the ensemble is

$$\boxed{H = H_{syst} + H_{env} + H_{int}}$$

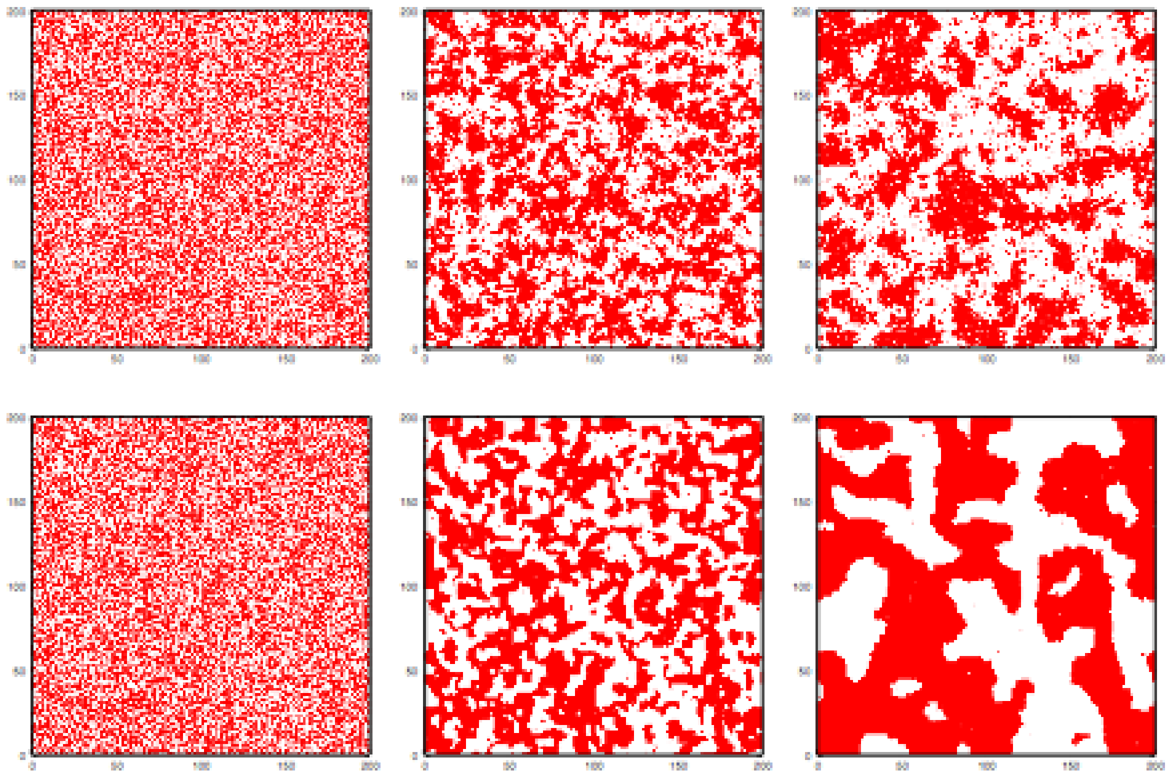


The dynamics of all variables are given by Newton (or Heisenberg) rules, depending on the variables being classical (or quantum).

$\mathcal{E}_{syst}(t) \neq$ ct, and $e_0 \ll \mathcal{E}_{syst} \ll \mathcal{E}_{env}$.

$2d$ Ising model

Snapshots after an instantaneous quench to T at $t = 0$



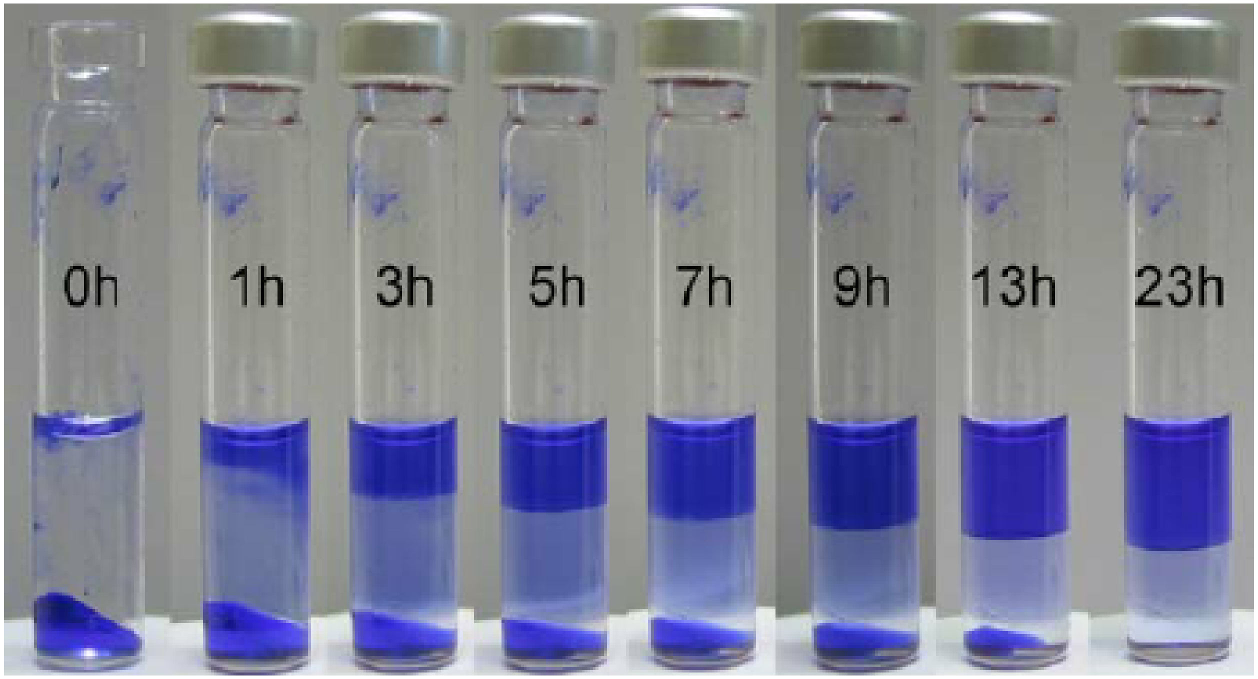
$T = T_c$

$T < T_c$

At $T = T_c$ **critical dynamics** At $T < T_c$ **coarsening**

A certain number of interfaces or domain walls in the last snapshots.

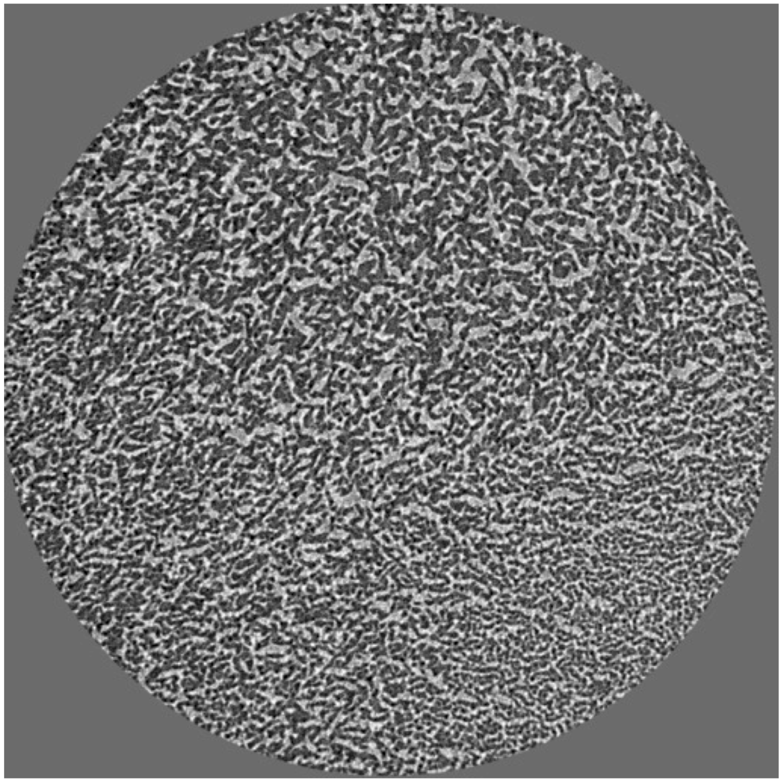
Membranes Proteins



Wadsten, Wöhri, Snijder, Katona, Gardiner, Cogdell, Neutze, Engström,

Lipidic Sponge Phase Crystallization of Membrane Proteins, J. Mol. Biol. 06

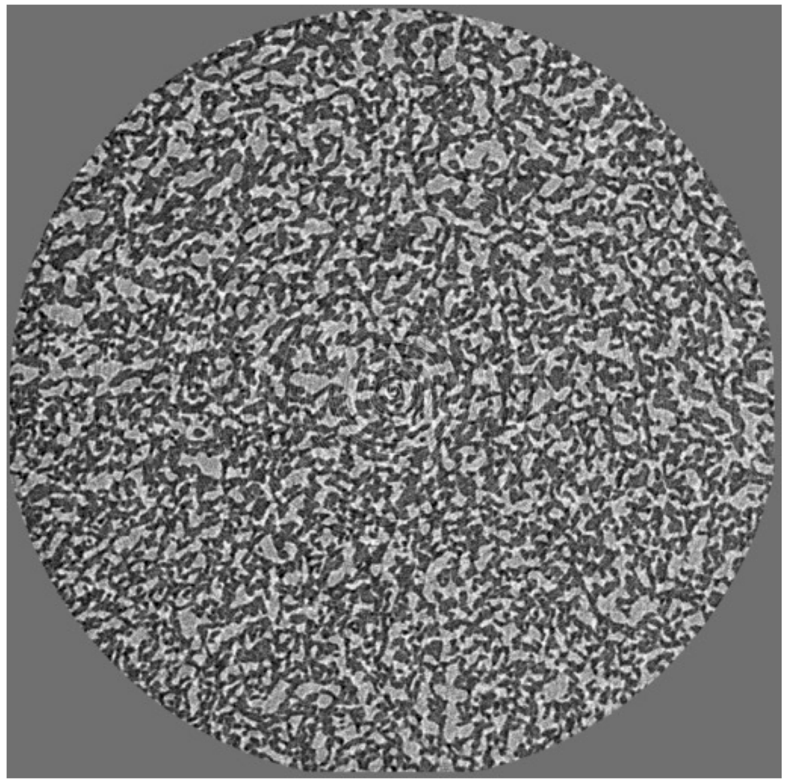
Phase separation in glasses



$t = 1$ min

Gouillart (Saint-Gobain), Bouttes & Vandembroucq (ESPCI) 11-14

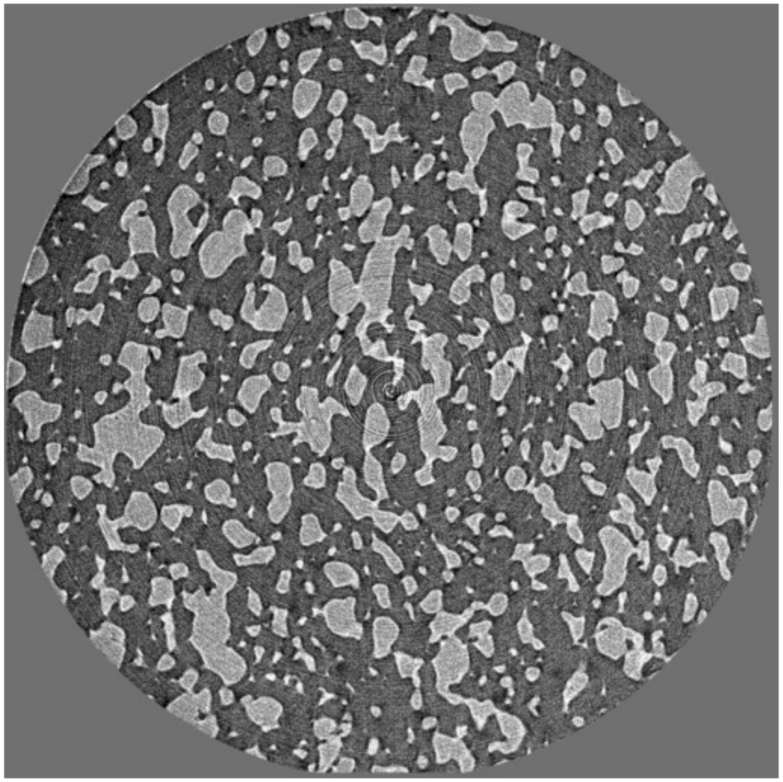
Phase separation in glasses



$t = 4$ min

Gouillart (Saint-Gobain), Bouttes & Vandembroucq (ESPCI) 11-14

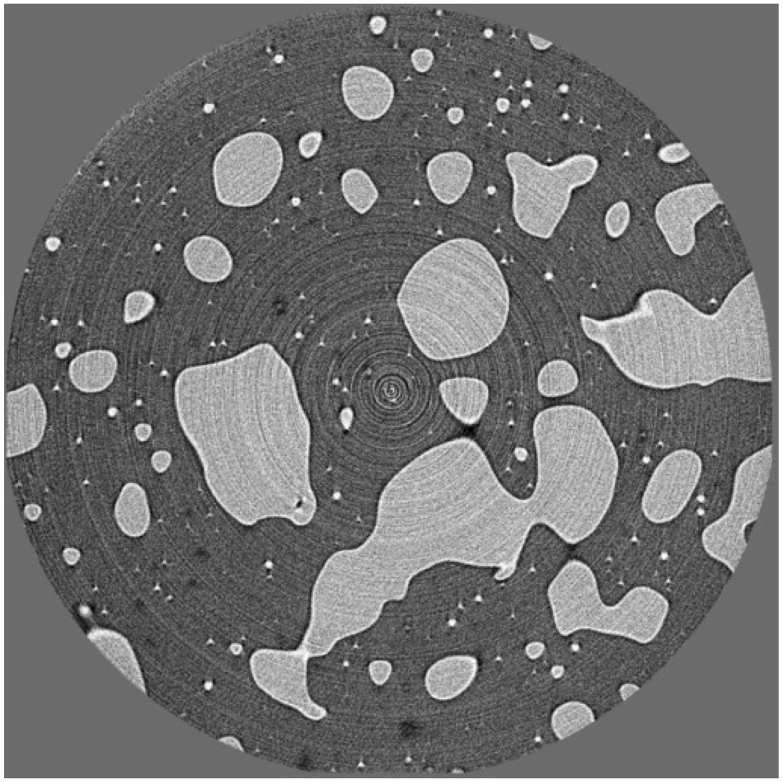
Phase separation in glasses



$t = 16$ min

Gouillart (Saint-Gobain), Bouttes & Vandembroucq (ESPCI) 11-14

Phase separation in glasses



$t = 64$ min

Gouillart (Saint-Gobain), Bouttes & Vandembroucq (ESPCI) 11-14

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$2d$ Ising Model (IM)

Archetypical example for classical magnetic systems

The Hamiltonian or energy function is

$$H = -J \sum_{\langle ij \rangle} s_i s_j$$

$s_i = \pm 1$ Ising spins.

$\langle ij \rangle$ sum over nearest-neighbours on the lattice.

$J > 0$ ferromagnetic coupling constant.

critical temperature $T_c > 0$ for $d > 1$.

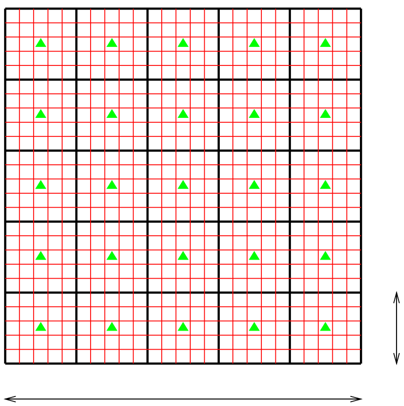
Equilibrium Paramagnetic & ferromagnetic states above & below T_c

Ginzburg-Landau

Continuous scalar statistical field theory

Coarse-grain the spin

$\phi(\vec{r}) = V_{\vec{r}}^{-1} \sum_{i \in V_{\vec{r}}} s_i$.



The partition function is $\mathcal{Z} = \int \mathcal{D}\phi \, e^{-\beta \mathcal{F}(\phi)}$ with

$$\boxed{\mathcal{F}(\phi) = \int d^d r \ \left\{ \frac{1}{2} [\nabla \phi(\vec{r})]^2 + \frac{T-J}{2} \phi^2(\vec{r}) + \frac{g}{4} \phi^4(\vec{r}) \right\}}$$

Elastic + potential energy with the latter inspired by the results for the fully-connected model (entropy around $\phi \sim 0$ and symmetry arguments).

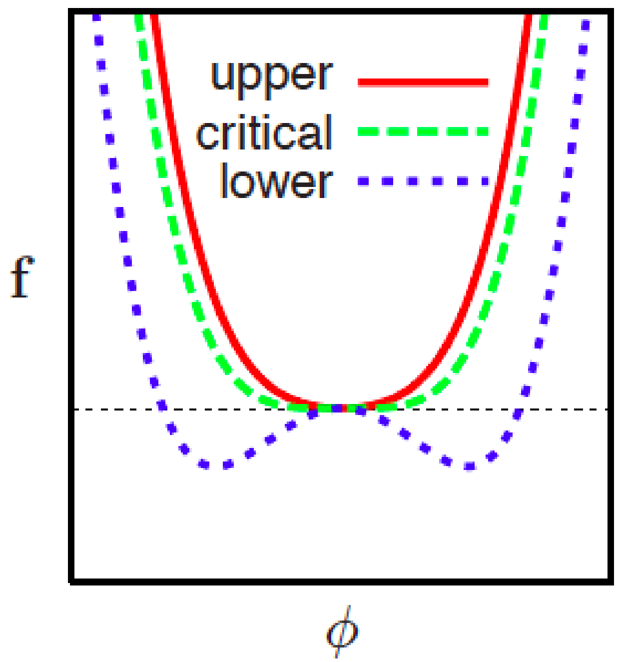
Uniform saddle point in the $V \to \infty$ limit : $\phi_{sp}(\vec{r}) = \langle \phi(\vec{r}) \rangle = \phi_0$.

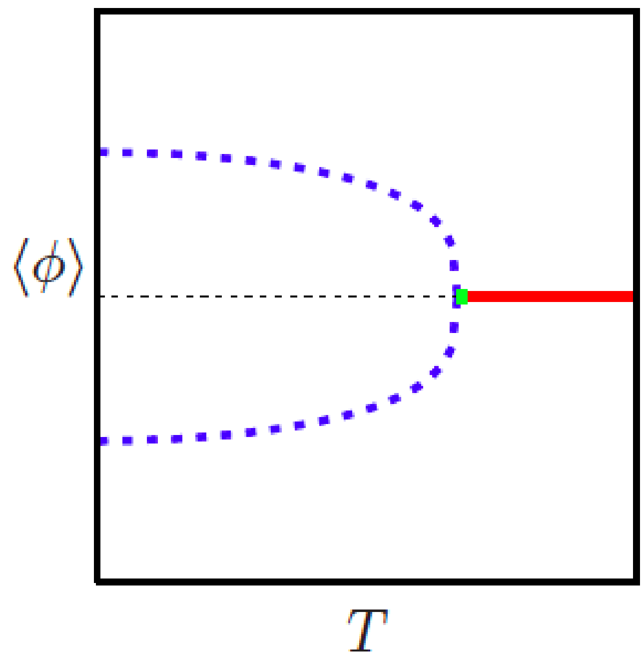
The free-energy density is $\lim_{V \to \infty} f_V(\beta, J, g) = \lim_{V \to \infty} V^{-1} \mathcal{F}(\phi_0)$.

2nd order phase-transition

Bi-valued equilibrium states related by symmetry

$\mathcal{F}$





Ginzburg-Landau free-energy

Scalar order parameter

e.g. Ising magnets

$2d$ Ising Model, dynamics

Archetypical example for classical magnetic systems

$$H = -J \sum_{\langle ij \rangle} s_i s_j$$

$s_i = \pm 1$ Ising spins.

$\langle ij \rangle$ sum over nearest-neighbours on the lattice.

$J > 0$ ferromagnetic coupling constant.

critical temperature $T_c > 0$ for $d > 1$.

Monte Carlo rule $s_i \to -s_i$ accepted with

$$p = 1 \quad \text{if} \quad \Delta\mathcal{E} < 0$$
$$p = e^{-\beta \Delta E} \quad \text{if} \quad \Delta\mathcal{E} > 0$$
$$p = 1/2 \quad \text{if} \quad \Delta\mathcal{E} = 0$$

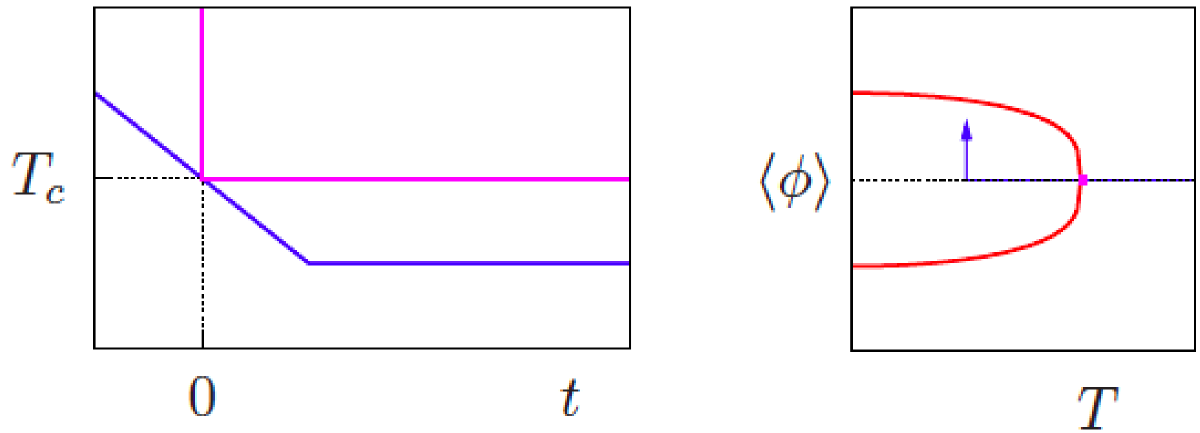
Non-conserved order parameter dynamics [$\uparrow\downarrow$ towards $\uparrow\uparrow$] etc. allowed.

[$m = 0$ to $m = 2$]

Evolution

Non-conserved order parameter dynamics

$$T(t) = T_c(1 - t/\tau_a)$$



Non-conserved order parameter $\langle\phi\rangle(t,T) \neq$ ct

e.g. single spin flips with Glauber or Monte Carlo stochastic rules.

Development of magnetization in a ferromagnet.

d-dimensional magnets

More general ferromagnetic models

$$H = -\sum_{\langle ij \rangle} J_{ij} \, \vec{s}_i \cdot \vec{s}_j$$

$J_{ij} > 0$	From a pdf with positive support
$\sum_{\langle ij \rangle}$	Sum over nearest-neighbours on a d-dim. lattice.
$s_i = \pm 1$	Ising spins.
$\vec{s}_i = (s_i^x, s_i^y)$	xy two-component spins.
$\ell^d \vec{\phi}(\vec{r}) = \sum_{i \in V_{\vec{r}}} \vec{s}_i$	Coarse-grained field over the volume $V = \ell^d$
L	Linear size of the system $L \gg \ell \gg a$
$T_c > 0$	for $d > 1$ and $L \to \infty$.

Coupling to the bath mimicked by Monte Carlo updates

Stochastic dynamics

Open systems

- **Microscopic**: identify the 'smallest' relevant variables in the problem (*e.g.*, the spins) and propose stochastic updates for them, as the Monte Carlo or Glauber rules.

- **Coarse-grained**: write down a stochastic differential equation for the field, such as the effective (Markov) Langevin equation

$$\underbrace{m\ddot{\vec{\phi}}(\vec{r},t)}_{\text{Inertia}} + \underbrace{\gamma_0\dot{\vec{\phi}}(\vec{r},t)}_{\text{Dissipation}} = \underbrace{\vec{F}(\vec{\phi})}_{\text{Deterministic}} + \underbrace{\vec{\xi}(\vec{r},t)}_{\text{Noise}}$$

with $\vec{F}(\vec{\phi}) = -\delta\mathcal{F}(\vec{\phi})/\delta\vec{\phi}$ (with the double-well $\mathbf{f}$)

e.g., time-dependent stochastic scalar Ginzburg-Landau equation or the stochastic Gross-Pitaevskii equation.

Models

Discrete *vs.* continuous

Ising spin models	Field theories
$H = -\sum_{ij} J_{ij} s_i s_j$	$\mathcal{F}[\phi] = \int d^d r \left[\frac{1}{2}(\nabla\phi)^2 - \frac{\mu}{2}\phi^2 + \frac{g}{4}\phi^4)\right]$
NCOP [$\uparrow\downarrow \mapsto \uparrow\uparrow$]	$\partial_t \phi(\vec{r},t) = \delta_{\phi(\vec{r},t)} \mathcal{F}[\phi] + \xi(\vec{r},t)$
COP [$\uparrow\downarrow \mapsto \downarrow\uparrow$]	$\partial_t \phi(\vec{r},t) \;\; = \;\; \nabla^2 \delta_{\phi(\vec{r},t)} \mathcal{F}[\phi] \; + \; \eta(\vec{r},t)$

Overdamped limit is fine

In the COP case $\langle \eta(\vec{x},t)\eta(\vec{y},t') \rangle = 2k_B T \nabla^2 \delta(\vec{x}-\vec{y})\delta(t-t')$

And generalisations for vector cases. **Quenched disorder** can be introduced by taking the J_{ij} or the parameters in the field theory from a pdf.

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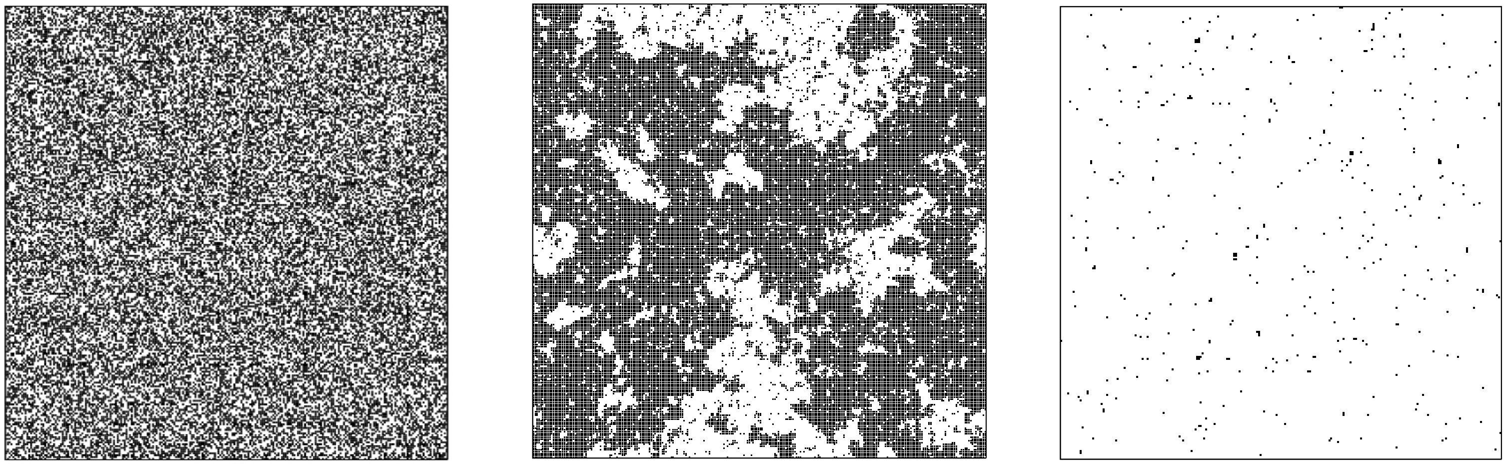
Equilibrium configurations

Up & down spins in a $2d$ Ising model

$T \to \infty$ $\quad$ $T = T_c$ $\quad$ $T < T_c$



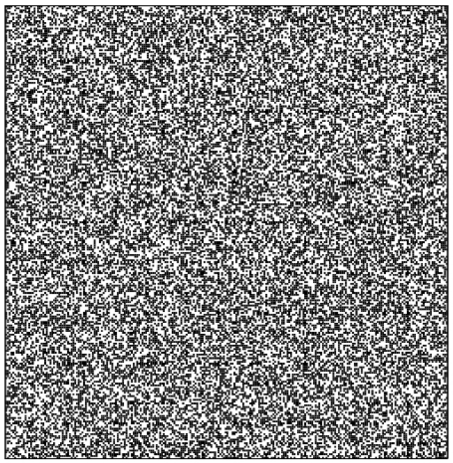
$\langle s_i \rangle_{eq} = 0$ $\quad$ $\langle s_i \rangle_{eq} = 0$ $\quad$ $\langle s_i \rangle_{eq^+} > 0$

$\langle \phi(\vec{r}) \rangle_{eq} = 0$ $\quad$ $\langle \phi(\vec{r}) \rangle_{eq} = 0$ $\quad$ $\langle \phi(\vec{r}) \rangle_{eq^+} > 0$

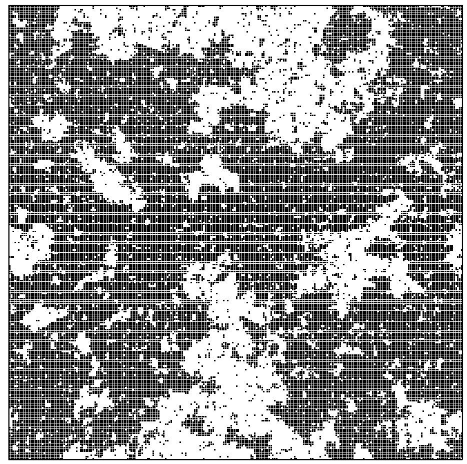
Coarse-grained scalar field $\phi(\vec{r}) \equiv \frac{1}{V_{\vec{r}}} \sum_{i \in V_{\vec{r}}} s_i$

The problem

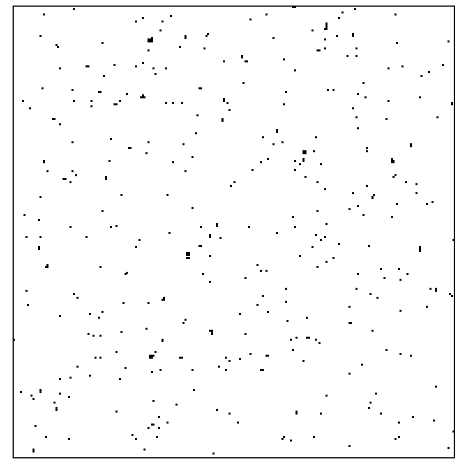
Up & down spins in a $2d$ Ising model



$T \to \infty$



$T = T_c$

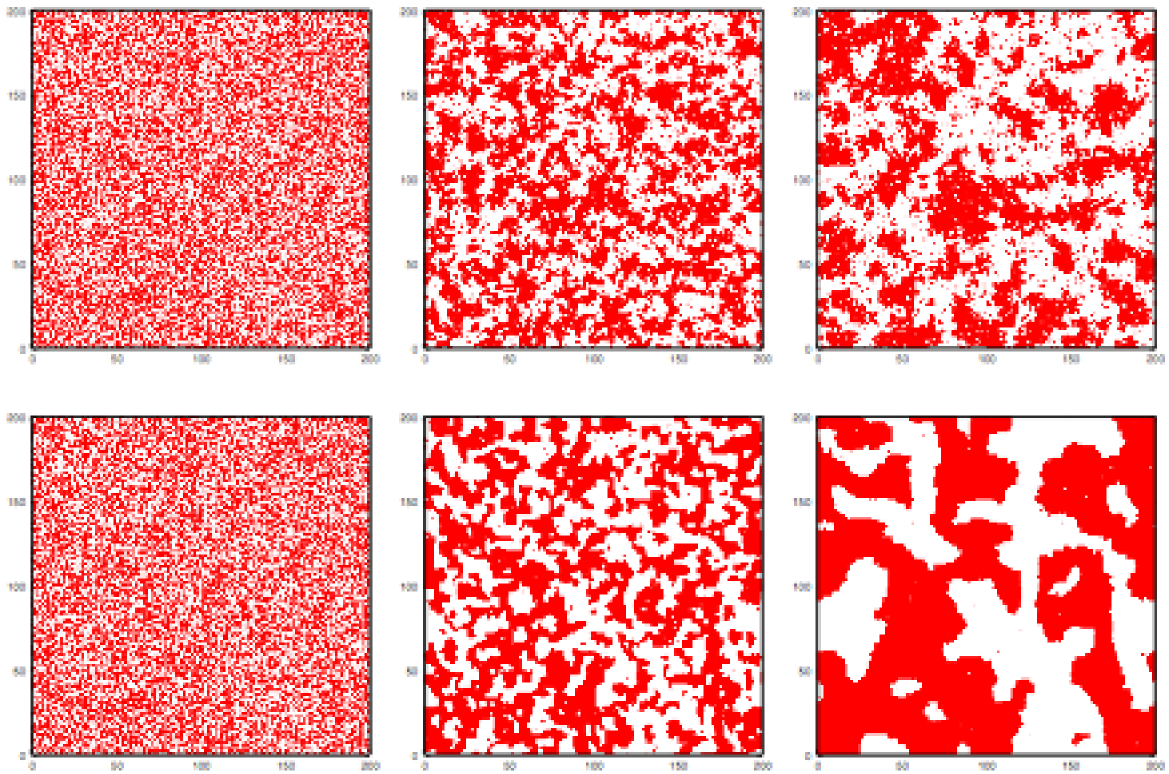


$T < T_c$

Question : starting from equilibrium at $T_0 \to \infty$ or $T_0 = T_c$ how is equilibrium at $T = T_c$ or $T < T_c$ attained ?

$2d$ Ising model

Snapshots after an instantaneous quench at $t = 0$



$T = T_c$

$T < T_c$

At $T = T_c$ **critical dynamics** At $T < T_c$ **coarsening**

A certain number of interfaces or domain walls in the last snapshots.

Domain growth

- At $T = T_c$ the system needs to grow structures of all sizes.

  Critical coarsening.

- At $T < T_c$: the system tries to order locally in one of the two competing equilibrium states at the new conditions.

  Sub-critical coarsening.

  The **linear size of the equilibrated patches increases in time**.

— The relaxation time t_r needed to reach $\langle \phi \rangle_{eq}(T/J)$ diverges with the size of the system, $t_r(T/J, L) \to \infty$ when $L \to \infty$ for $T \leq T_c$.

— Dissipative dynamics $d\langle \mathcal{E} \rangle / dt < 0$. Energy density is reduced by diminishing the density of **domain walls**.

Statement

In both cases one sees the growth of ‘red and white’ patches and

interfaces surrounding such geometric domains.

More precisely, spatial regions of local equilibrium (with vanishing or non-vanishing order parameter) grow in time and

a growing length $\mathcal{R}(t, T/J)$ can be computed with the help of dynamic scaling.

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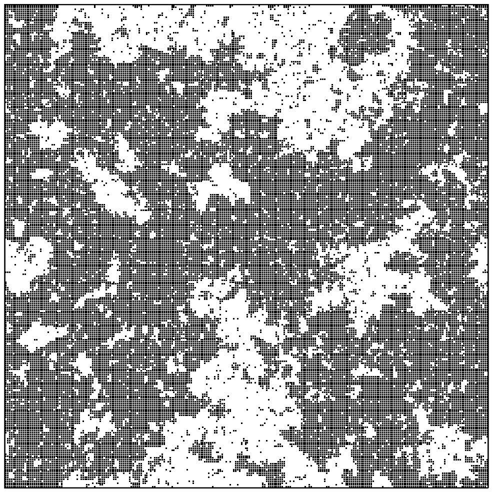
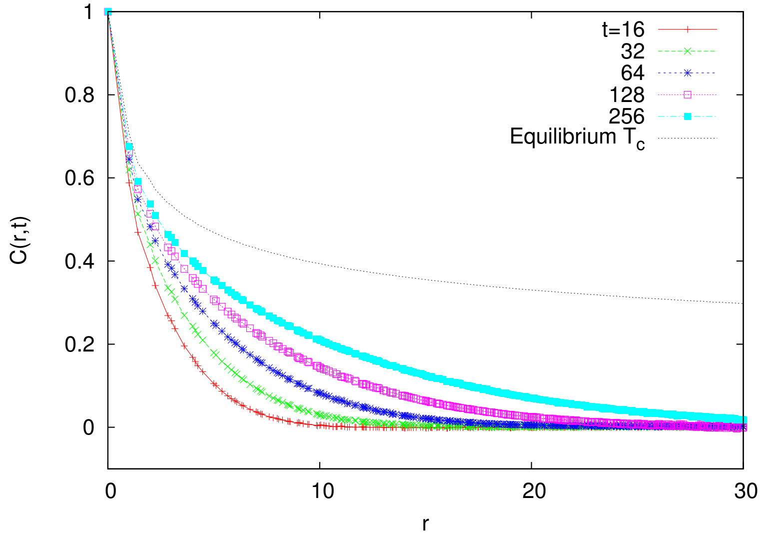
Critical growth





- Black curve : equilibrium relaxation, $r^{2-d-\eta}$.

- Coloured curves are for different times after the quench and they slowly approach the equilibrium one.

- From $C(\mathcal{R}_c(t), t) = 1/e$ one gets $\boxed{\mathcal{R}_c(t) \simeq t^{1/z_{eq}}}$

  (Other prescriptions give equivalent results.)

Dynamic scaling

After quenches (set $J = 1$ for notational simplicity)

At late times there is a single length-scale, the typical radius of the equilibrium structures (domains below T_c) $\mathcal{R}(t,T)$, such that the structure is (in statistical sense) independent of time when lengths are scaled by $\mathcal{R}(t,T)$, e.g.

$$C(r,t) \equiv \langle \, s_i(t) s_j(t) \, \rangle|_{|\vec{x}_i - \vec{x}_j| = r} \sim \; f\left(\frac{r}{\mathcal{R}(t,T)} \right) ,$$

$$C(t,t_w) \equiv \langle \, s_i(t) s_i(t_w) \, \rangle \sim f_c \left(\frac{\mathcal{R}(t,T)}{\mathcal{R}(t_w,T)} \right) ,$$

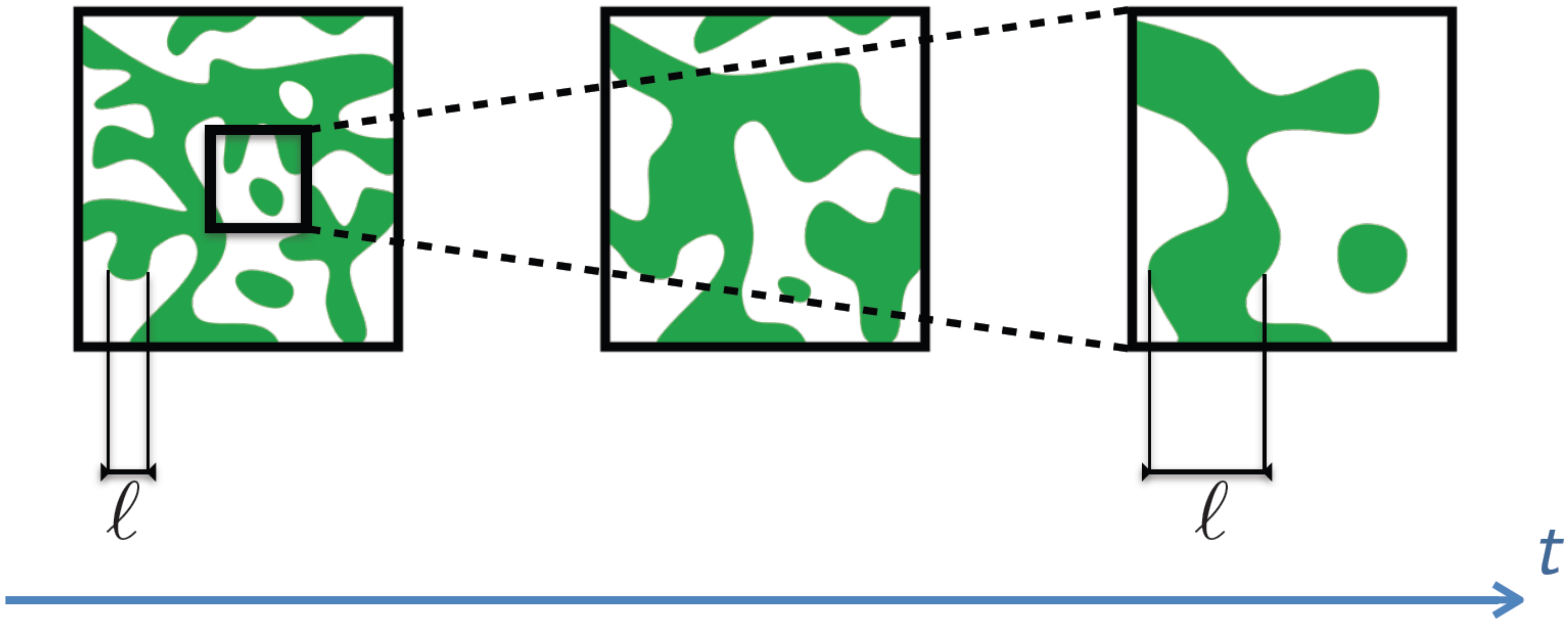
etc. when $L \gg r \gg \xi(T)$, $t, t_w \gg t_0$ and C small enough (see below).

Suggested by experiments and numerical simulations. Proved for a few cases.

Review Bray, 1994.

Dynamic scaling

in phase ordering kinetics



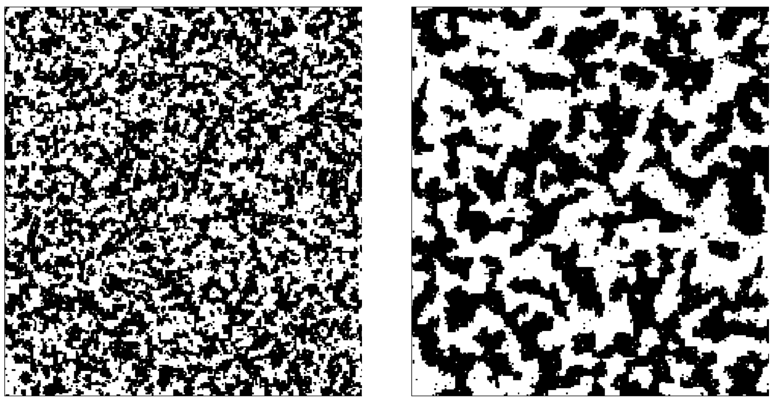
Growing length $\ell(t)$ and equilibrium reached for $\ell(t_{eq}) \simeq L$

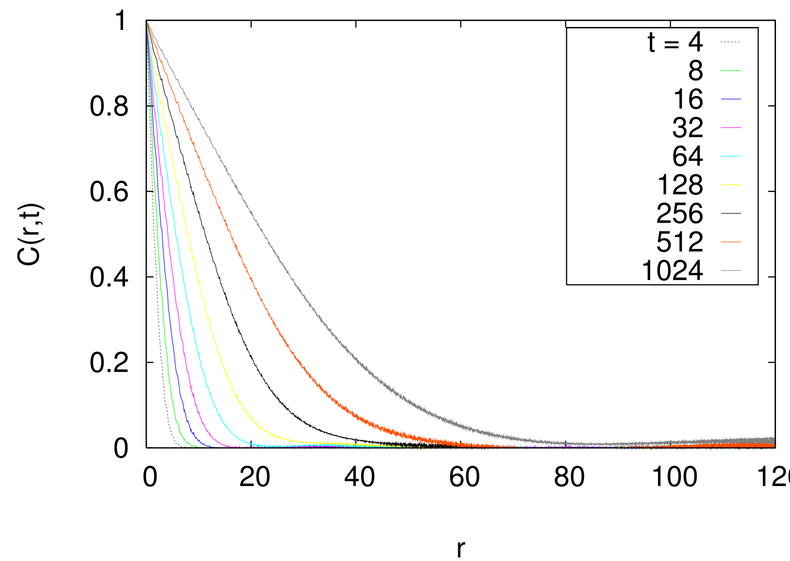
Typically $\ell(t) \simeq t^{1/z}$ and $t_{eq} \simeq L^z$

Excess energy w.r.t. the equilibrium one stored in the domain walls ; $\Delta\mathcal{E}(t) \simeq \ell^{-a}(t)$

Dynamic scaling

Quench of the $2d$IM with NCOP from $T_0 \to \infty$ to $T = 0$





$$\langle \phi(t) \rangle = 0 \qquad C^c_{eq}(r) \simeq e^{-r/\xi_{eq}}$$

- Coloured curves are $C(r,t)$ for different times after the quench.
- The growing length is $\boxed{\mathcal{R}(t,T) \simeq t^{1/z_d}}$ with $\boxed{z_d = 2}$
- $\mathcal{R}(t,T)$ is the averaged linear size of the domains.

Review Bray 94

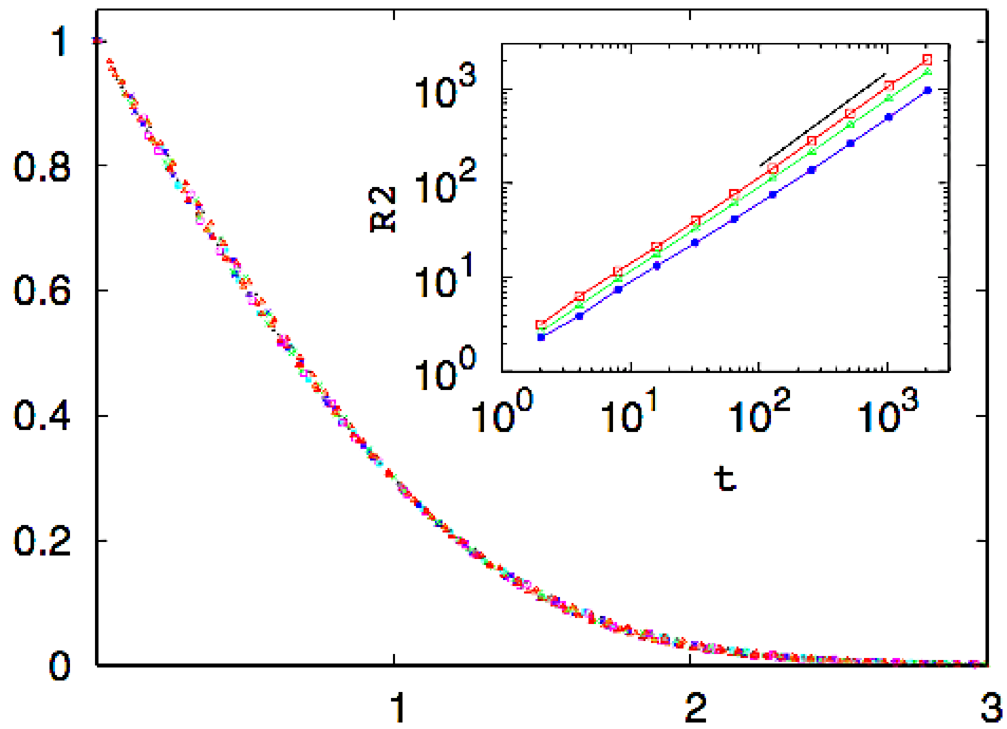
Dynamic scaling

Quench of the $2d$IM with NCOP from $T_0 \to \infty$ to $T \simeq 0$

Scaling regime $a \ll r \ll L$, $r \simeq \mathcal{R}(t,T) \simeq t^{1/z_d}$

$$C(r,t) \simeq m_{eq}^2(T)\, f_c\left(\frac{r}{\mathcal{R}(t,T)}\right)$$

Scaling looks perfect



$r/\mathcal{R}(t,T)$

Space-time correlation

Separation of fime-scales & dynamic scaling

Critical quench

$$C(r,t) \simeq C_{eq}(r)\, f_c\left(\frac{r}{\mathcal{R}_c(t)}\right)$$

$C_{eq}(r) \simeq r^{2-d-\eta}$, $\lim_{x\to 0} f_c(x) = 1$ and $\lim_{x\to\infty} f_c(x) = 0$.

Sub-critical

$$C(r,t) \simeq [C_{eq}(r) - m_{eq}^2] + m_{eq}^2\, f\left(\frac{r}{\mathcal{R}(t,T)}\right)$$

$C(0,t) = 1\ \forall t$, $\lim_{r\to 0} C_{eq}(r) = 1$, $\lim_{r\to\infty} C_{eq}(r) \propto \langle s_i \rangle_{eq}^2 = m_{eq}^2$, $\lim_{x\to 0} f(x) = 1$ (long times) and $\lim_{x\to\infty} f(x) = 0$ (short distances).

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Growing length

Dynamic universality classes

Use the growing length $\mathcal{R}(t, T)$ to identify dynamic universality classes.

They depend on the dimension of the order parameter and the dynamic mechanism of growth (intimately related to the conservation laws).

Growing length

Dynamic universality classes at the critical point

At T_c, dynamic RG techniques work very well.

U. C. Tauber, *Critical Dynamics : A Field Theory Approach to Equilibrium and Non-Equilibrium Scaling Behavior* (Cambridge University Press, 2014)

One finds dynamic scaling, with the growing length

$$\mathcal{R}_c(t) \simeq t^{1/z_c}$$

$\boxed{z_c}$ can be computed with methods that are very similar to critical exponents in static phase transitions.

Dynamic universality classes classified by the z_c values.

The scaling functions can be estimated as well

Growing length

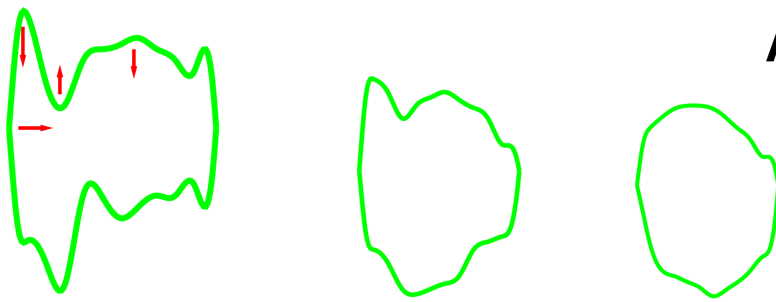
Dynamic universality classes below the critical point

No systematic method

Focus on the dynamic mechanisms

Scalar field w/NCOP dynamics

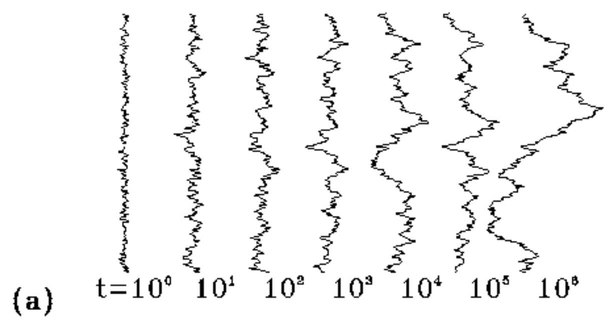
- Curvature driven ($T = 0$): $\vec{v} \equiv \dfrac{d\vec{n}}{dt} \propto K\hat{n}$ with $K = \vec{\nabla} \cdot \hat{n}$



Allen & Cahn 79

- Domain wall roughening ($T > 0$)

- Domain wall roughening and pinning by quenched disorder



e.g. elastic line in random media. **Kolton** *et al* **05**

Curvature driven

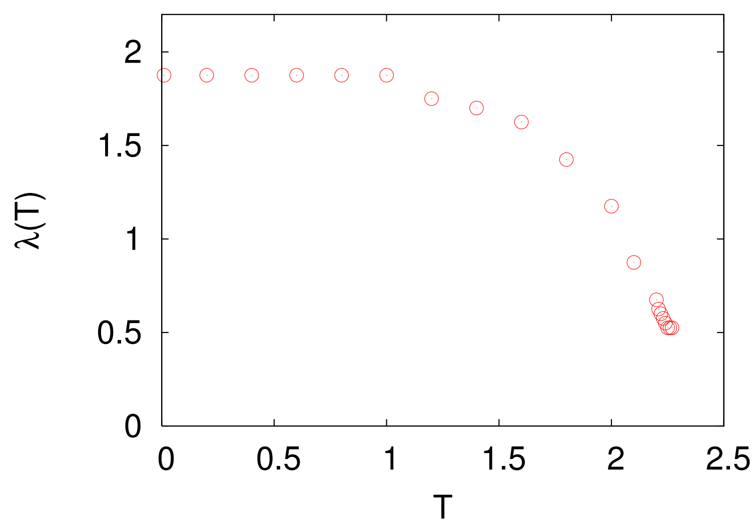
Numerical solution of the time-dependent Ginzburg-Landau equation

A. Langins (1st year master project)

MC dynamics 2dIM

The typical length-scale $\Leftrightarrow$ a typical area

$$\mathcal{R}(t,T) \sim \lambda(T)\, t^{1/2} \qquad \Leftrightarrow \qquad \mathcal{A}(t,T) \sim \lambda^2(T)\, t$$



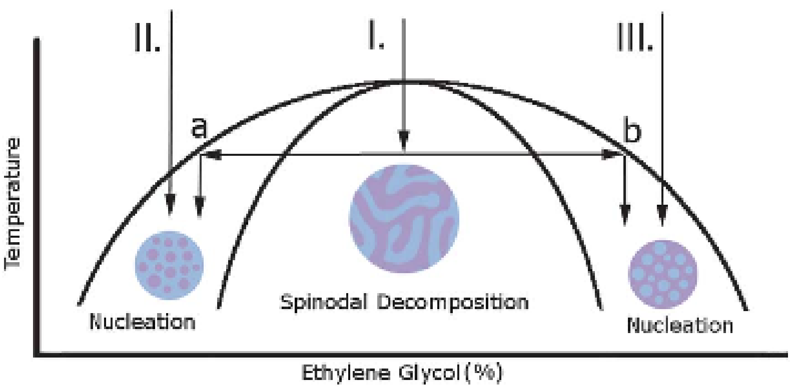
NB the exponent $\frac{1}{2}$ is independent of T and the details of the dynamics, lattice, *etc.* as long as the order parameter is non-conserved & there is no disorder.

The T-dependence in $\lambda(T)$ is due to the roughening of the domain walls.

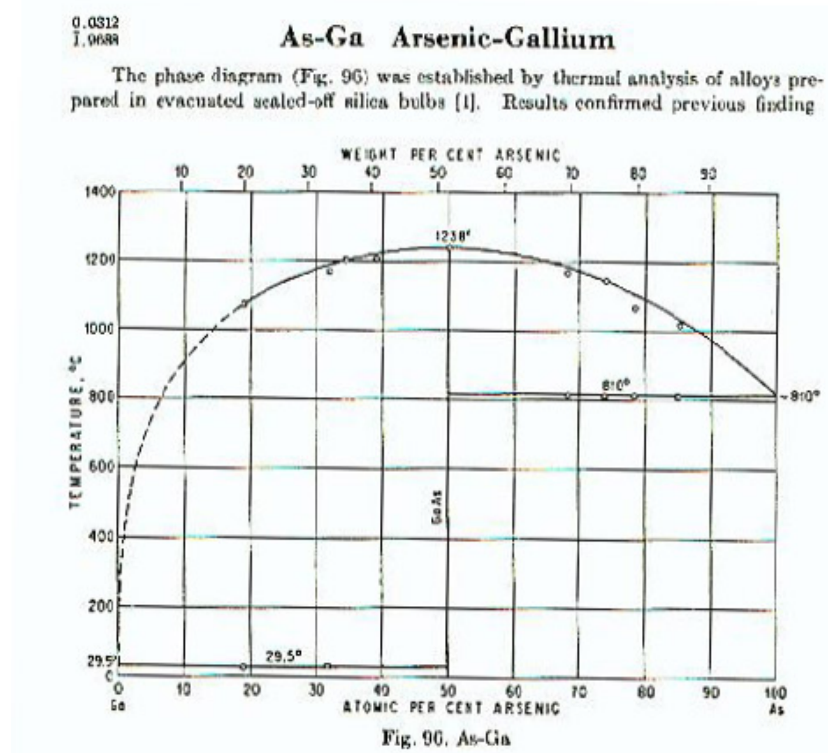
Phase separation

Demixing transitions

Two species ● and ●, repulsive interactions between them.



Sketch



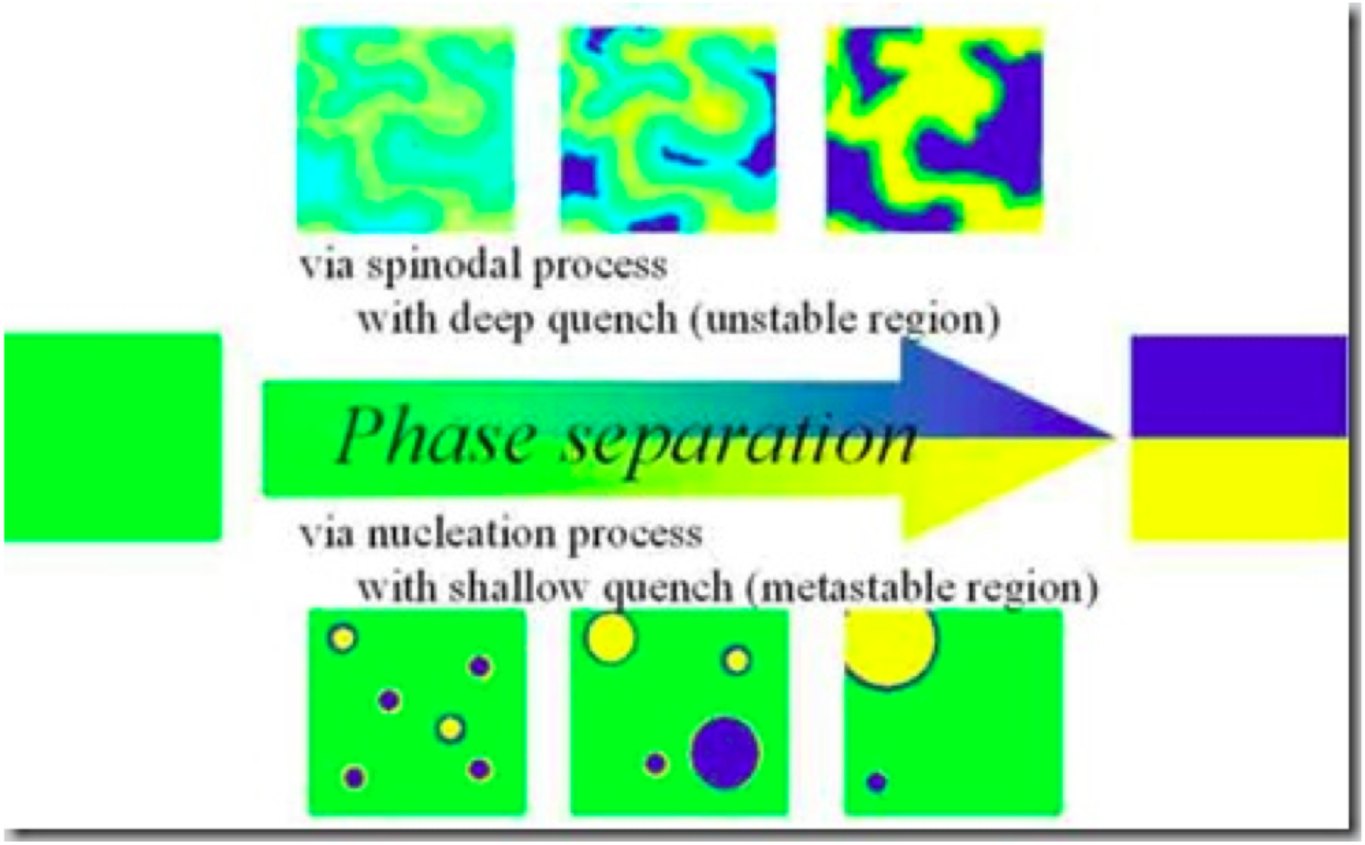
Experimental phase diagram

Binary alloy, **Hansen & Anderko, 54**

Scalar field w/COP

Phase separation

Matter diffusion



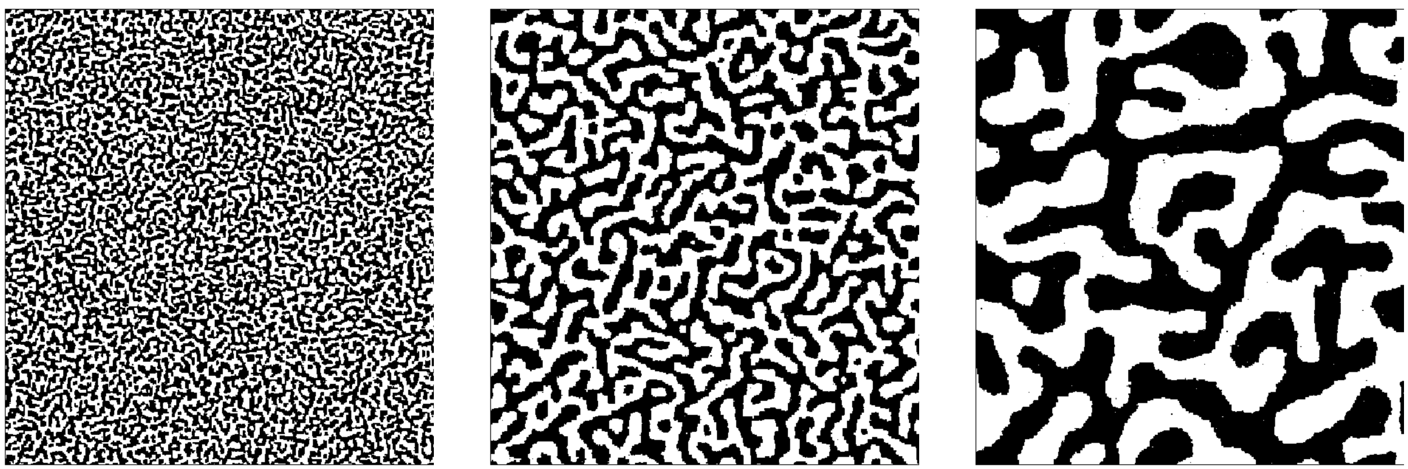
Phase separation

Spinodal decomposition in binary mixtures

A species $\equiv$ spin up ; B species $\equiv$ spin down

$2d$ Ising model with Kawasaki dynamics at T

locally conserved order parameter



50 : 50 composition Rounder boundaries

$$\mathcal{R}(t,T) \simeq \lambda(T) t^{1/3}$$

Huse 93

Weak disorder

e.g., random ferromagnets

At short time scales the dynamics is relatively fast and independent of the quenched disorder ; thus

$$\mathcal{R}(t,T) \simeq \lambda(T) t^{1/z_d}$$

At longer time scales domain-wall pinning by disorder dominates.

Assume there is a length-dependent barrier $B(\mathcal{R}) \simeq \Upsilon \mathcal{R}^\psi$ to overcome

The Arrhenius time needed to go over such a barrier is $t_A \simeq t_0 \, e^{\frac{B(\mathcal{R})}{k_B T}}$

This implies

$$\boxed{\mathcal{R}(t,T) \simeq \left(\frac{k_B T}{\Upsilon} \ln t/t_0 \right)^{1/\psi}}$$

Weak disorder

Still two ferromagnetic states related by symmetry

$$\mathcal{R}(t,T) \simeq \begin{cases} \lambda(T) t^{1/z_d} & \mathcal{R} \ll L_c(T) \quad \text{curvature-driven} \\ L_c(T)(\ln t/t_0)^{1/\psi} & \mathcal{R} \gg L_c(T) \quad \text{activated} \end{cases}$$

with $L_c(T)$ a growing function of T.

Inverting times as a function of length $\quad t \simeq [\mathcal{R}/\lambda(T)]^{z_d}\; e^{\mathcal{R}/L_c(T)}$

At short times this equation can be approximated by an effective power law with a T-dependent exponent :

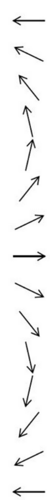
$$t \simeq \mathcal{R}^{\overline{z}_d(T)} \qquad\qquad \overline{z}_d(T) \simeq z_d\,[1 + \text{ct}/L_c(T)]$$

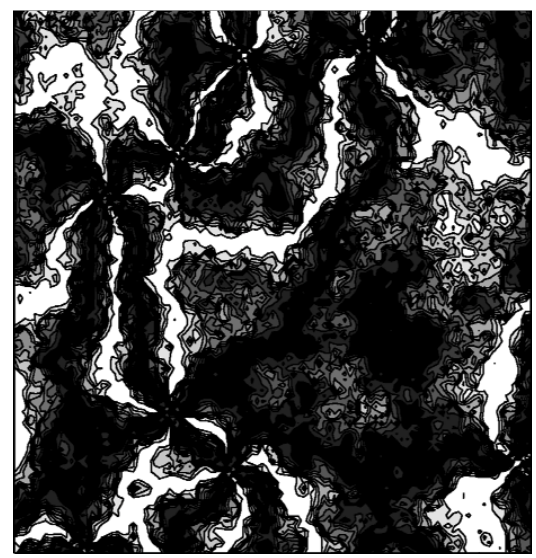
Fisher & Huse, Paul, Rieger & Schehr, etc. Crossover in **Bustingorry et al. 09**

Planar magnets

Schrielen pattern : gray scale according to $\sin^2 2\theta_i(t)$

Spin-waves Vortices (planar spins turn around these points)





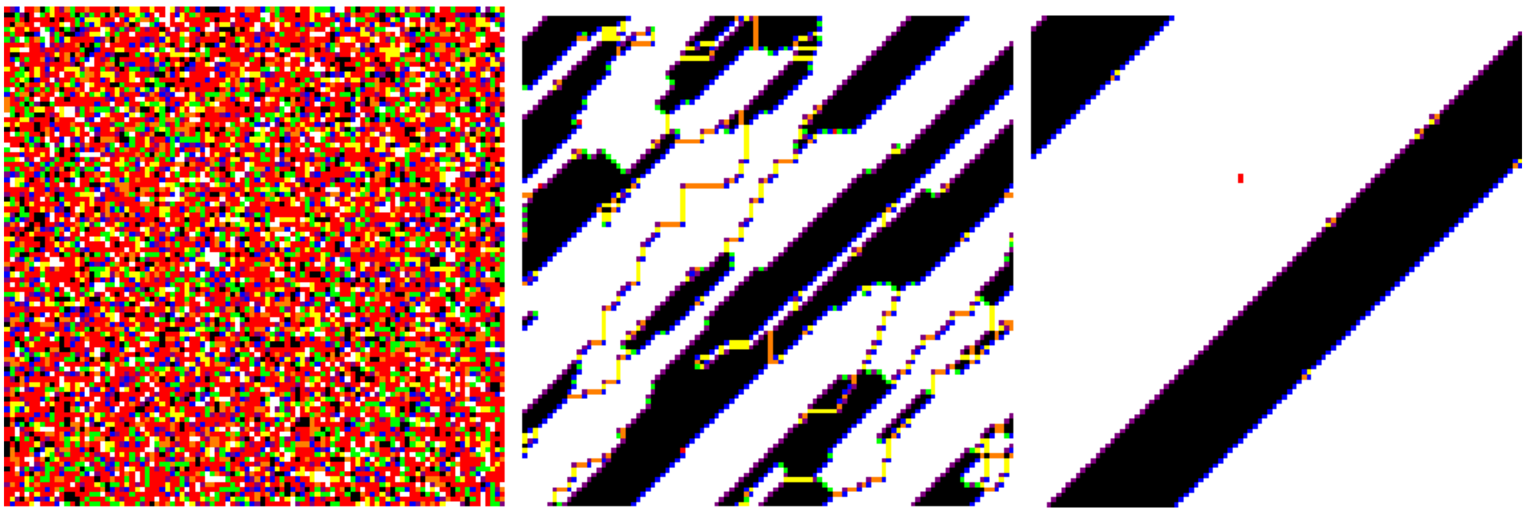
After a quench vortices annihilate and tend to bind in pairs

$$\mathcal{R}(t,T) \simeq \lambda(T)\,[t/\ln(t/t_0(T))]^{1/2}$$

Yurke ***et al*** **93, Bray & Rutenberg 94, Jelic & LFC 11**

Frustrated magnets

e.g., $2d$ spin ice or vertex models



Stripe growth in the FM phase

Anisotropic growth, $\mathcal{R}_\perp(t,T)$ and $\mathcal{R}_\parallel(t,T)$

Levis & LFC 11

Universality classes

as classified by the growing length

$$\mathcal{R}(t,T) \simeq \begin{cases} \lambda(T)\, t^{1/2} & \text{scalar NCOP} \quad z_d = 2 \\ \lambda(T)\, t^{1/3} & \text{scalar COP} \quad z_d = 3 \\ \lambda(T) \left(\dfrac{t}{\ln t}\right)^{1/2} & \text{planar NCOP in} \quad d = 2 \\ \text{etc.} & \end{cases}$$

Temperature and other microscopic parameters appear in the prefactor.

Universality classes

as classified by the growing length

$$\mathcal{R}(t,T) \simeq \begin{cases} \lambda(T)\, t^{1/2} & \text{scalar NCOP} \quad z_d = 2 \\ \lambda(T)\, t^{1/3} & \text{scalar COP} \quad z_d = 3 \\ \lambda(T) \left(\dfrac{t}{\ln t}\right)^{1/2} & \text{planar NCOP in} \quad d = 2 \\ \lambda(T)\, (\ln t)^{1/\psi} & \text{weak disorder NCOP} \end{cases}$$

Are scaling functions independent of

temperature, other parameters, microscopic dynamics ?

Super-universality ?

Review Bray 94, Corberi, Lippiello, Mukherjee, Puri & Zannetti 11

Plan of the lecture

1. The phenomenon
2. Theoretical setting
3. Critical and sub-critical quenches
4. **Dynamic scaling (again)**
5. Dynamic universality classes
6. Two-time correlations and ageing
7. Two-time responses and loss of memory
8. Mean-field models
9. Modern studies

Dynamic scaling

Scaling functions

very early MC simulations **Lebowitz et al 70s** & experiments

One identifies a **growing linear size of equilibrated patches**

$$\boxed{\mathcal{R}(t,T)}$$

If this is the only length governing the dynamics, the **space-time correlation functions** should scale with $\mathcal{R}(t,T)$ according to

At $T = T_c$ $\qquad C(r,t) \simeq C_{eq}(r)\, f_c(\frac{r}{\mathcal{R}_c(t)})$ $\qquad$ **Scaling fct f_c ✔**

At $T < T_c$ $\qquad C(r,t) \simeq C^c_{eq}(r) + m^2_{eq}\, f(\frac{r}{\mathcal{R}(t,T)})$ $\qquad$ **Scaling fct f ?**

Reviews **Hohenberg & Halperin 77** (critical) **Bray 94** (sub-critical)

Plan of the lecture

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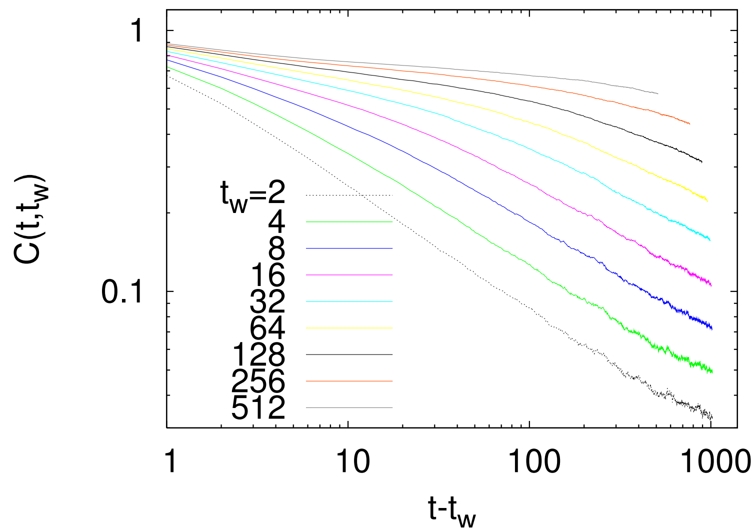
Two-time self-correlation

e.g., MC simulation of the $2d$IM at $T < T_c$

$$C(t,t_w) = N^{-1} \sum_{i=1}^{N} \langle s_i(t) s_i(t_w) \rangle$$



Stationary relaxation

Aging decay

Separation of time-scales : stationary – aging

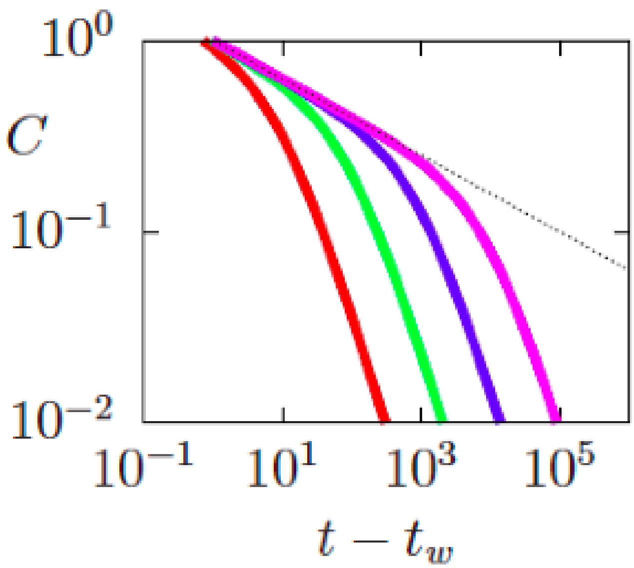
$$C(t,t_w) = C_{eq}(t-t_w) + m_{eq}^2 \, f\left(\frac{\mathcal{R}(t,T)}{\mathcal{R}(t_w,T)}\right)$$

$C_{st}(0) = 1 - m_{eq}^2$, $\lim_{x\to\infty} C_{st}(x) = 0$, $f(1) = 1$, $\lim_{x\to\infty} f(x) = 0$.

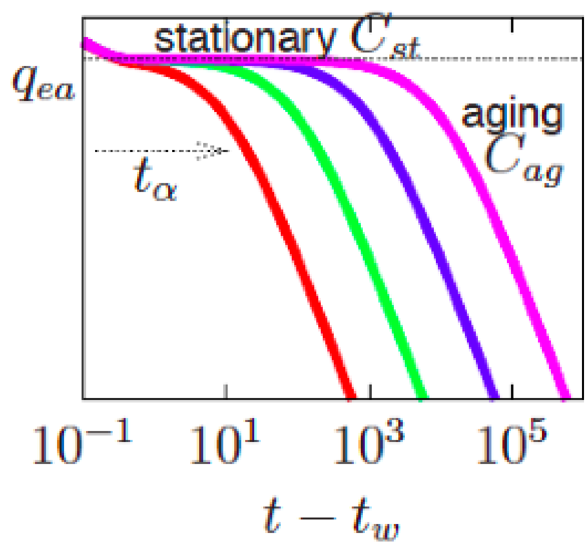
Two-time self-correlation

Comparison

Critical coarsening ($T = T_c$)



Sub-critical coarsening ($T < T_c$)



Separation of time-scales

Multiplicative

Additive

Aging

Older samples relax more slowly

Older samples need more time to relax

spontaneously (correlation functions)

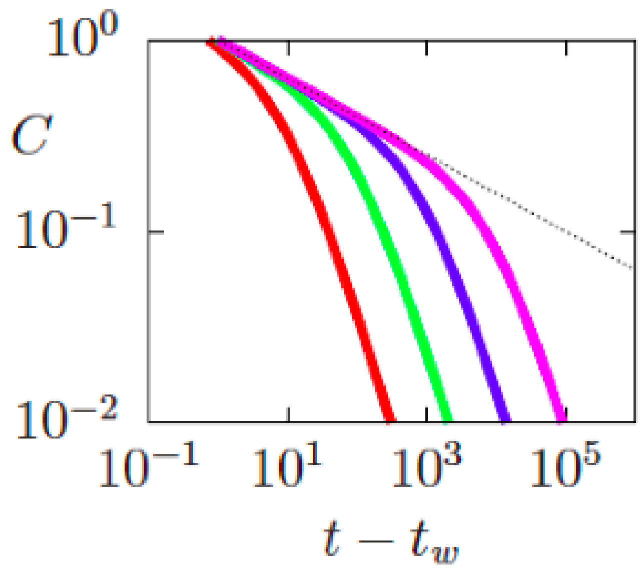
after a change in conditions (response functions)

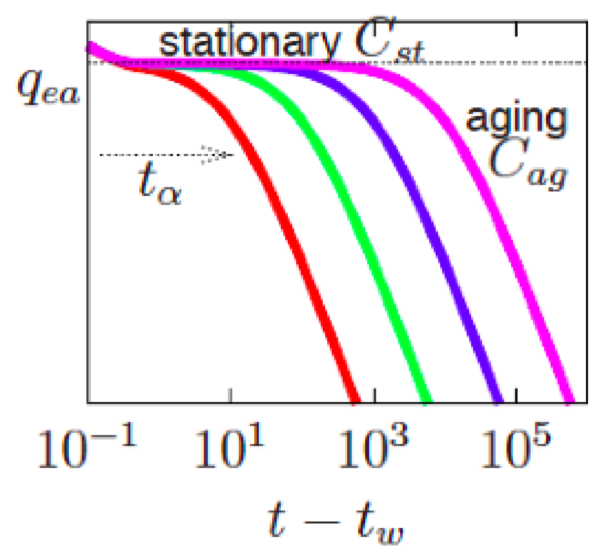
t_w is the time that measures the age of the system

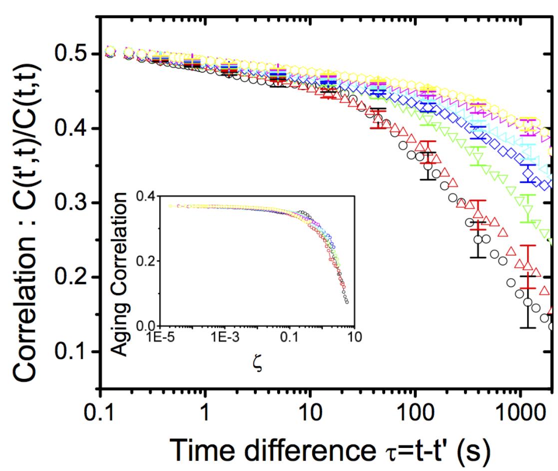
Huge literature on this phenomenology. Some reviews of experimental measurements were written by **Struick** on polymer glasses, **Vincent et al.** & **Nordblad et al.** on spin-glasses, **McKenna et al.** on all kinds of glasses.

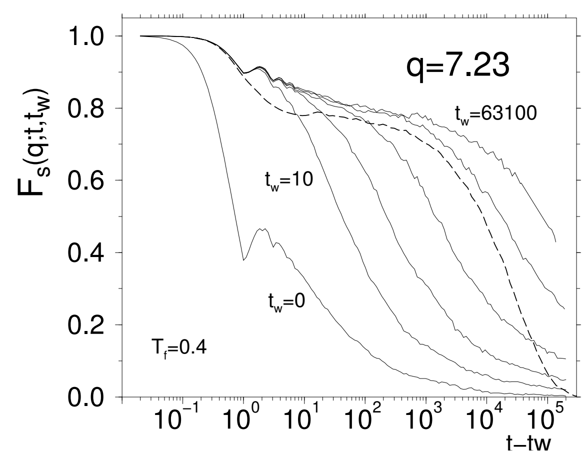
Two-time self-correlation

Comparison









Thiospinel (spin-glass)

Lennard-Jones mixture (glass)

Plan of the lecture

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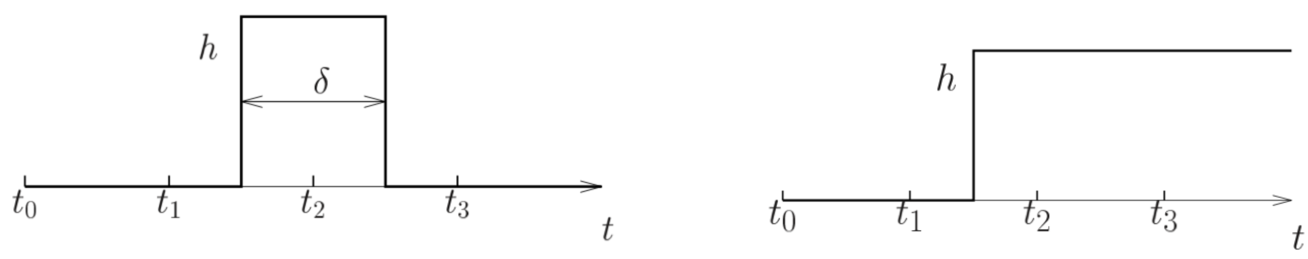
Response to perturbations



The perturbation couples linearly to the observable $B[\{s_i\}]$

$$H \quad \to \quad H - hB[\{s_i\}]$$

The linear instantaneous response of another observable $A[\{s_i\}]$ is

$$R_{AB}(t, t_w) \equiv \left\langle \left. \frac{\delta A[\{s_i\}](t)}{\delta h(t_w)} \right|_{h=0} \right\rangle$$

The linear integrated response or dc susceptibility is

$$\chi_{AB}(t, t_w) \equiv \int_{t_w}^{t} dt' \, R_{AB}(t, t')$$

Linear response

Critical and sub-critical coarsening

Critical coarsening

$$\chi(t,t_w) = \beta - \chi_{eq}(t-t_w) g\left(\frac{\mathcal{R}_c(t)}{\mathcal{R}_c(t_w)}\right)$$

Sub-critical coarsening

$$\chi(t,t_w) = \chi_{eq}(t-t_w) + [\mathcal{R}(t_w,T)]^{-a_\chi}\, g\left(\frac{\mathcal{R}(t,T)}{\mathcal{R}(t_w,T)}\right)$$

In both cases : $\chi_{eq}(t-t_w) = -(k_B T)^{-1} dC_{eq}(t-t_w)/d(t-t_w)$.

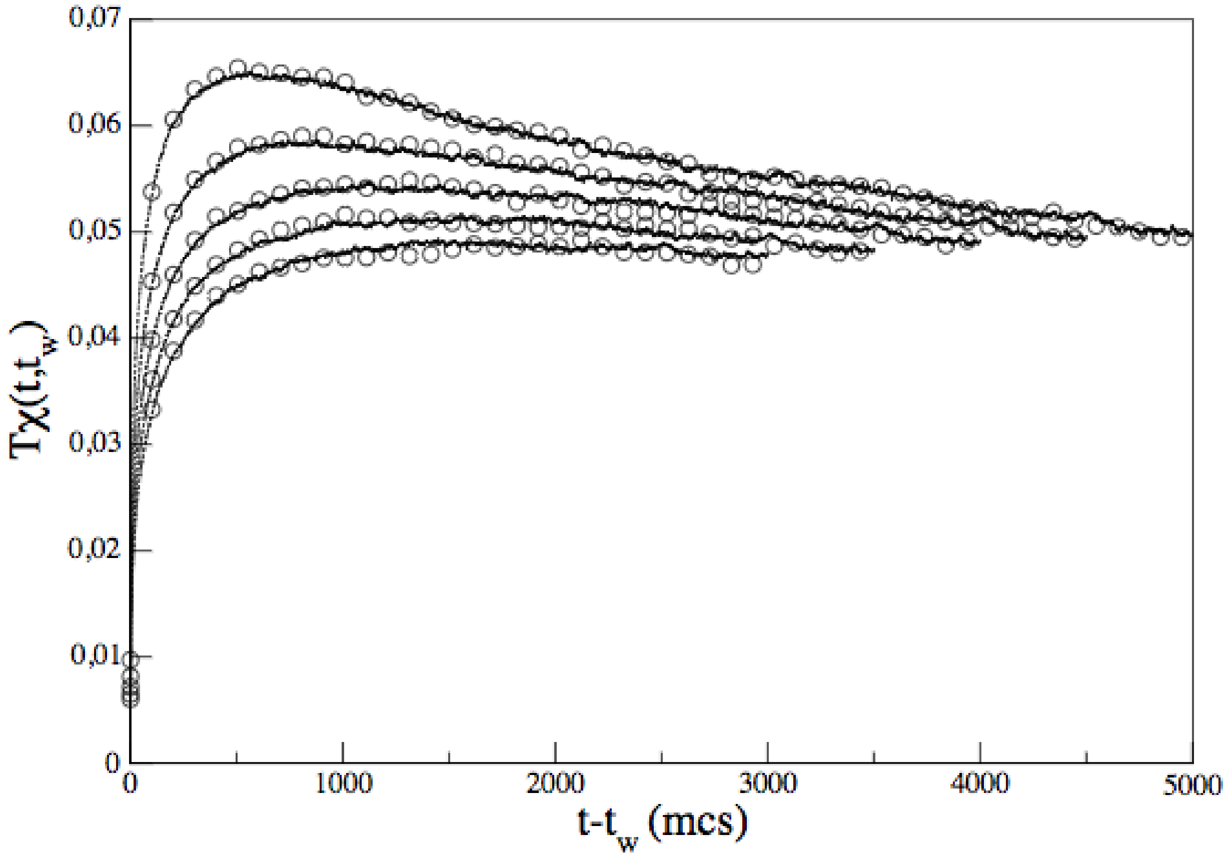
To be proven in the 3rd Lecture

Reviews

Crisanti & Ritort 03, Calabrese & Gambassi 05, Corberi *et al.* 07, LFC 11

Linear response

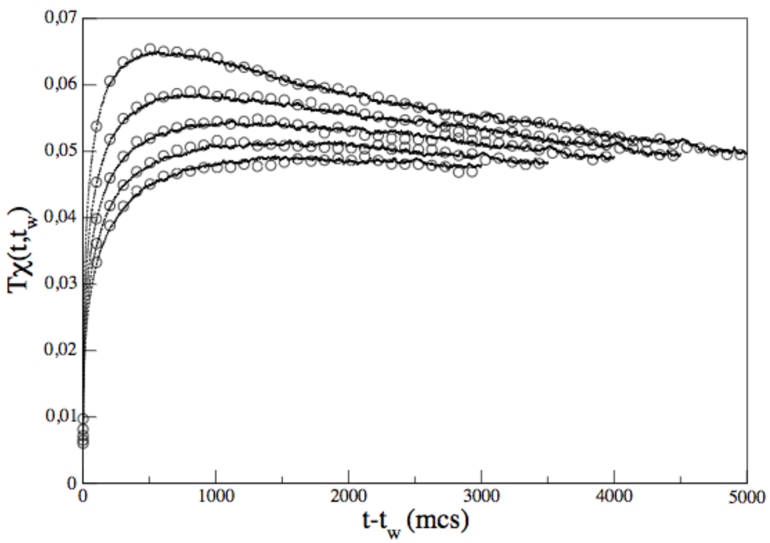
Sub-critical coarsening in the MC dynamics of 2dIM

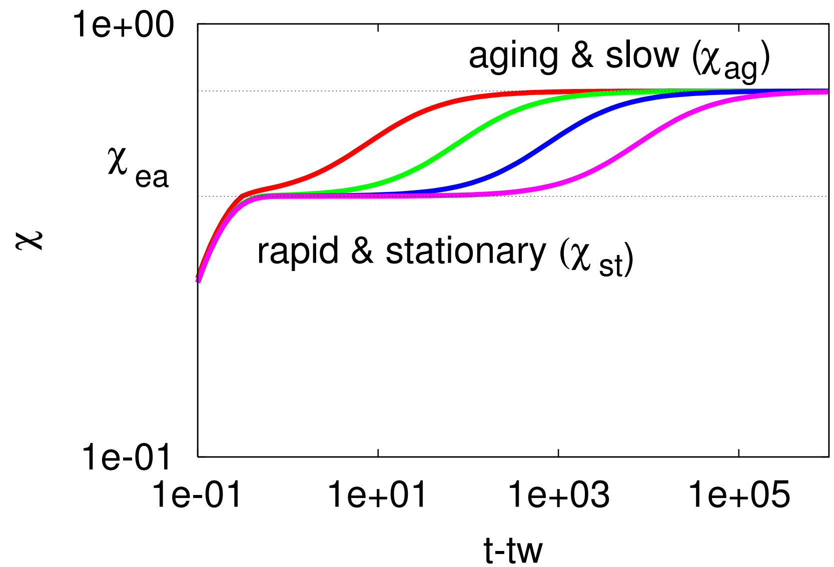


Lippiello, Corberi & Zannetti 05

Linear response

Coarsening vs glassy





Lippiello, Corberi & Zannetti 05

Sketch **Chamon & LFC 07**

There is no (weak) long-term memory in the coarsening problem. Just the stationary part will remain asymptotically, contrary to the sketch on the right for glasses & spin-glasses.

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The spherical $p = 2$ model

$$H = -\sum_{ij} J_{ij} s_i s_j + z \left(\sum_i s_i^2 - N \right)$$

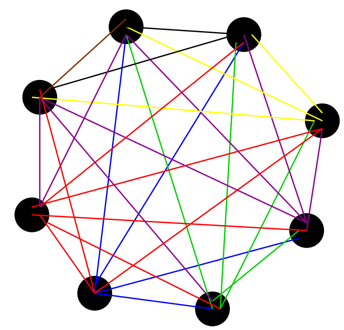
Fully connected interactions

Gaussian distributed

interaction strengths J_{ij}

Spherical spins $\sum_i s_i^2 = N$

z is a Lagrange multiplier



$$\rho(\lambda_\mu) \propto \sqrt{(2J)^2 - \lambda_\mu^2}$$

$$H = -\sum_{\mu} \lambda_\mu s_\mu^2 + z \left(\sum_\mu s_\mu^2 - N \right)$$

Key: the largest eigenvalue

becomes **diffusive**,

$\lambda_{\max} - z_\infty = 0$

Same scaling laws for two-time corr. and resp. but no space dependence

The O(N) model

Upgrade the field to a vector $\phi \mapsto \vec{\phi}$ with $a = 1, \ldots, N$ components

$$\vec{\phi} = (\phi_1, \ldots, \phi_N)$$

The (over-damped) Ginzburg-Landau equation is now

$$\gamma_0 \partial_t \phi_a(\vec{r}, t) = -\frac{\delta \mathcal{F}[\vec{\phi}]}{\delta \phi_a(\vec{r}, t)} + \xi_a(\vec{r}, t)$$

The $N \to \infty$ limit allows one to decouple the vector components :

$$\phi_a(\vec{r}, t)[\mu - \tfrac{1}{N} \textstyle\sum_{b=1}^{N} \phi_b^2(\vec{r}, t)] \mapsto \phi_a(\vec{r}, t) z(t)$$

and the equations are now linear with a global constraint.

Coarsening is linked to the growth of the **diffusive** $\vec{k} = 0$ mode.

The O(N) model

Upgrade the field to a vector $\phi \mapsto \vec{\phi}$ with $a = 1, \ldots, N$ components

$$\vec{\phi} = (\phi_1, \ldots, \phi_N)$$

The equations are now linear with a global constraint

$$\gamma_0 \partial_t \phi_a(\vec{r}, t) = \nabla^2 \phi_a(\vec{r}, t) + z(t)\phi_a(\vec{r}, t) + \xi_a(\vec{r}, t)$$

and

$$z(t) = \mu - N^{-1} \sum_a \phi_a^2(\vec{r}, t)$$

Solve for $\phi_a(\vec{r}, t)$ as a function of $z(t)$ and then impose the constraint to fix $z(t)$.

Coarsening is linked to the growth of the $\vec{k} = 0$ mode, i.e. tendency to homogeneous order.

Summary

- **At and below** T_c growth of equilibrium structures.
- The linear size of the equilibrium patches is measured by $\boxed{\mathcal{R}(t,T)}$
- At T_c vanishing order parameter

  Multiplicative scaling $\boxed{C \simeq C_{eq} C_{ag} \, ; \chi \simeq \chi_{eq} \chi_{ag}}$

- Below T_c non-vanishing order parameter

  Additive scaling $\boxed{C \simeq C_{eq} + C_{ag} \, ; \chi \simeq \chi_{eq} + \chi_{ag}}$

- In both cases C_{ag} is finite while χ_{ag} vanishes asymptotically.

We shall discuss χ and how it compares to C later.

Phase ordering kinetics

The lecture was about

- Growth of equilibrium patches at T_c and below T_c.
- Divergence of $t_{eq}(L)$ with the system size.
- Existence of a single growing length $\mathcal{R}(t,T)$
- Separation of time-scales and dynamic scaling, *e.g.* $C = C_{eq} + C_{ag}$.
- Two kinds of correlations : Space-time and two-time ones.
- Dynamic universality classes at and below T_c.
- The more tricky/rich linear susceptibility.

Is there a static growing length in all systems with slow dynamics ?

Which one ?

Plan of the lecture

1. The phenomenon
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Voter model

Similar questions can be asked in very well-known problems in math, *e.g.*

Dynamics of a voter model starting from

a random initial condition

- Purely dynamic, **no energy**, no equilibrium phase transition
- But, two absorbing states
- The **dynamic mechanism** towards absorption is understood

domain growth is driven by interfacial noise

$2d$ Voter Model

Archetypical example of opinion dynamics

H does not exist - kinetic model

$s_i = \pm 1$ Ising spins that

sit on the vertices of a lattice.

Voter update rule

choose a spin at random, say s_i

choose one of its $2d$ neighbours at random, say s_j

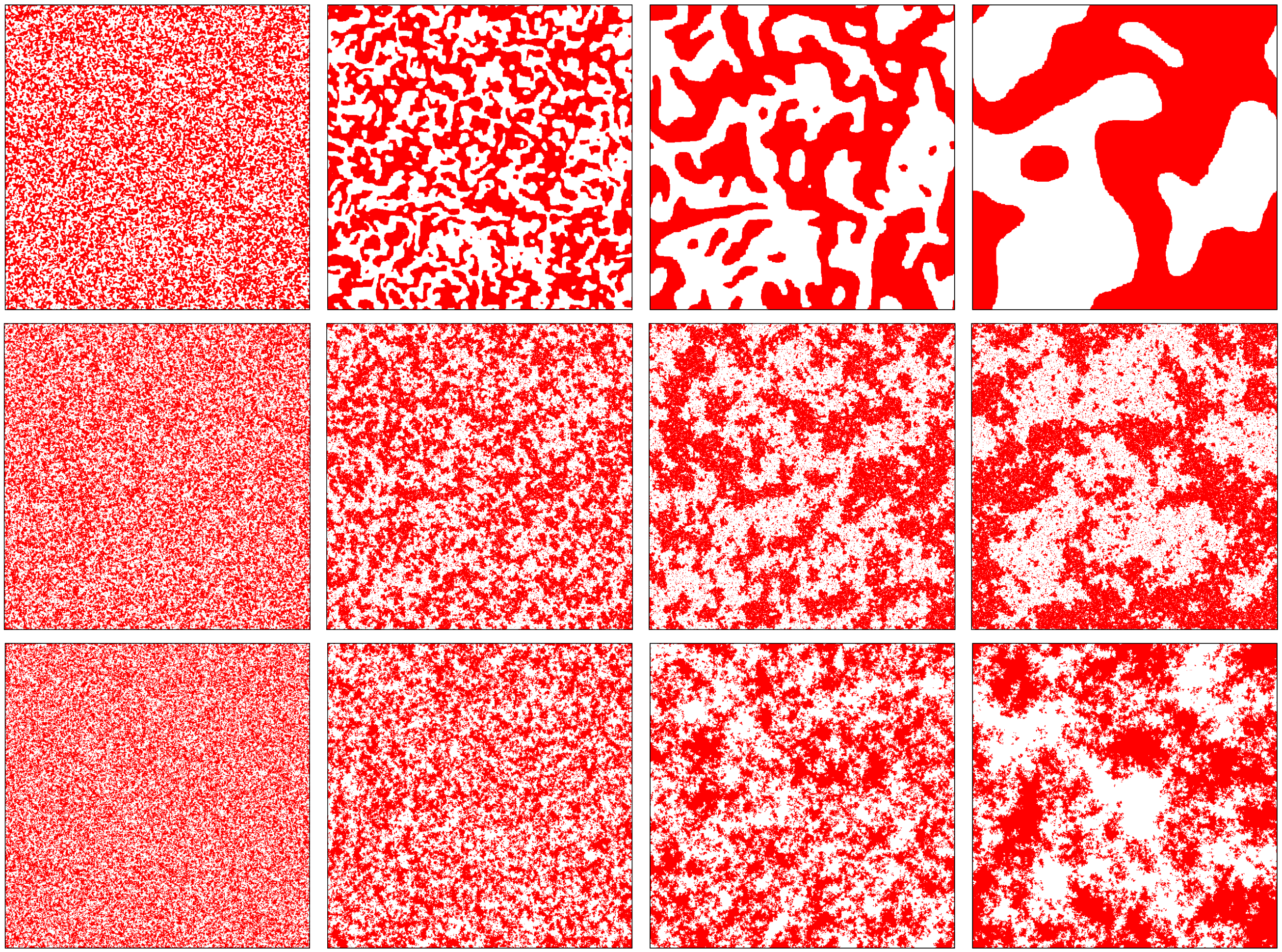
set $s_i = s_j$

In two dimensions full consensus, *i.e.* $m = L^{-d} \sum_{i=1}^{L^d} s_i = \pm 1$ is reached in a timescale $t_C \simeq L^2$ (with $\ln L$ corrections)

Clifford & Sudbury 73, Holley & Liggett 75, Cox & Griffeaths 86

Phase ordering kinetics

$s_i = \pm 1$ **at** $t = 0$ **MCs, snapshots at** $t = 4,\ 64,\ 512, 4096$ **MCs**



Ising

$T = 0$

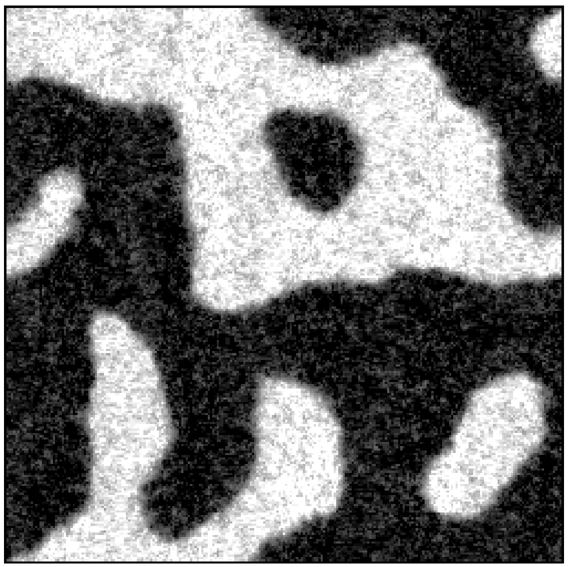
T_c

Voter

Multiplicative noise

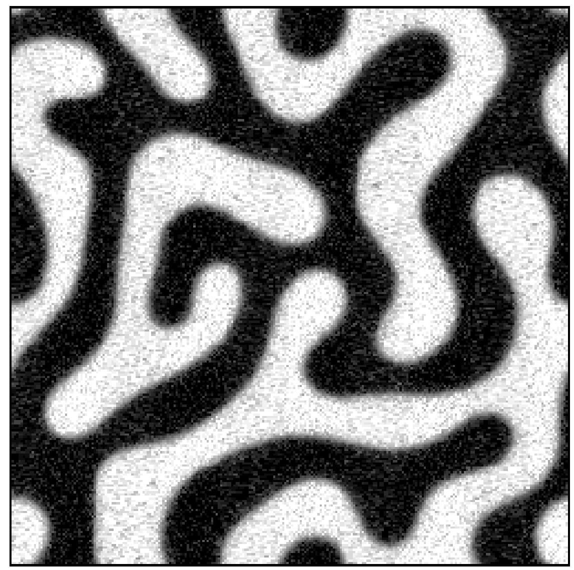
Noise induced coarsening for $\partial_t \phi = \nabla^2 \phi - \mu\phi - g\phi^3 - \phi\xi$

NCOP



$\mathcal{R}(t,T) \simeq t^{1/2}$

COP

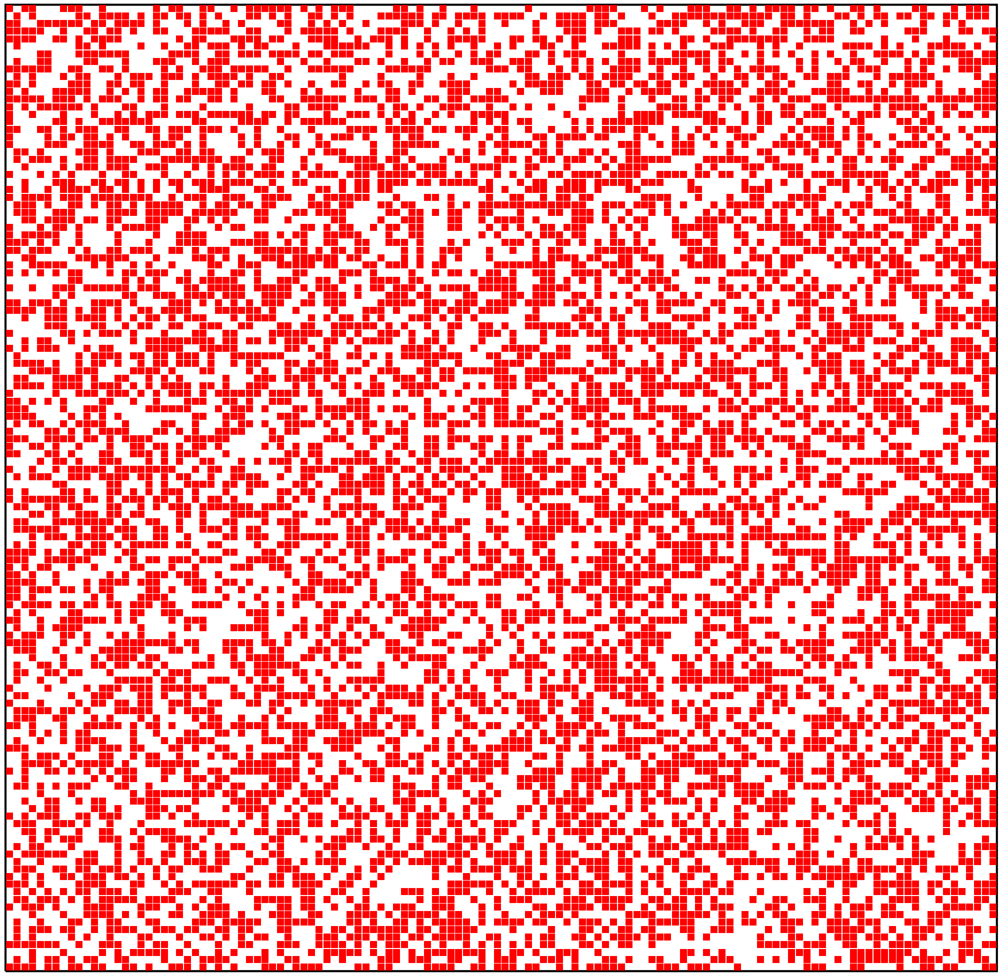


$\mathcal{R}(t,T) \simeq t^{1/3}$

Ibañes, García-Ojalvo, Toral & Sancho 00

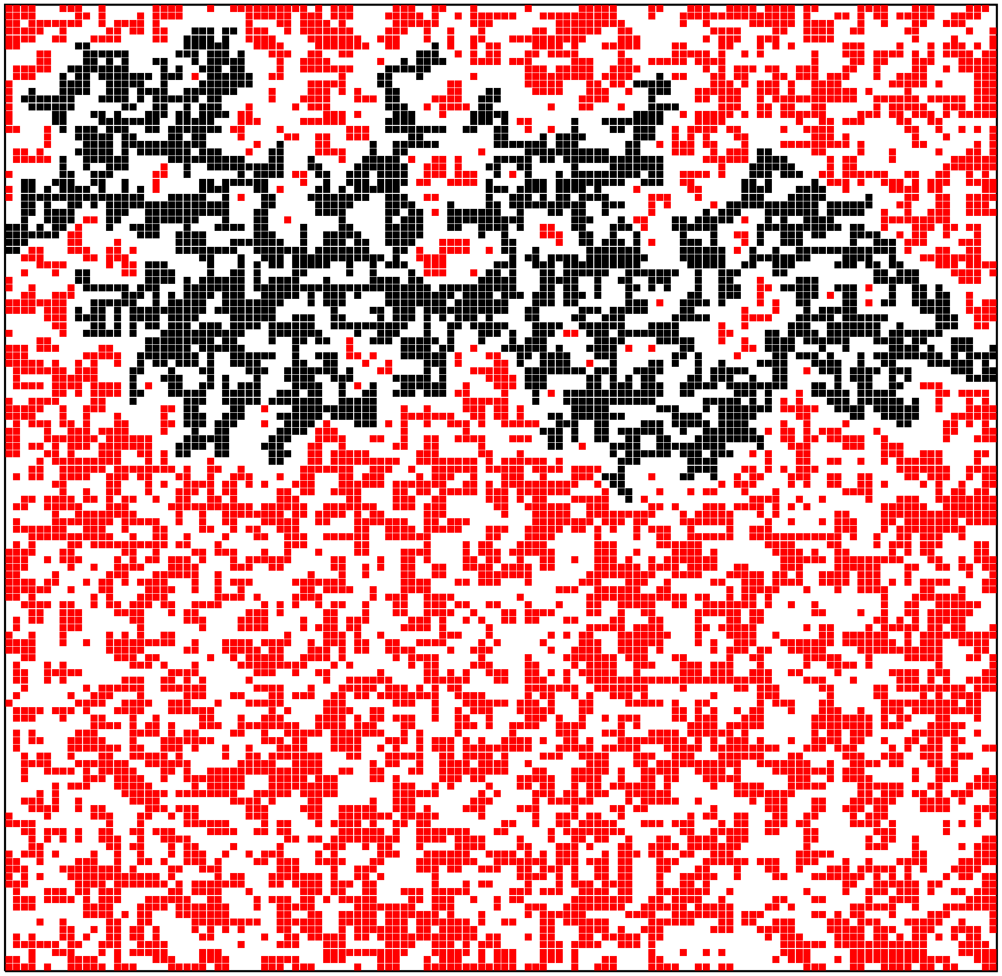
Percolation issues

2d square IM quenched to T=0



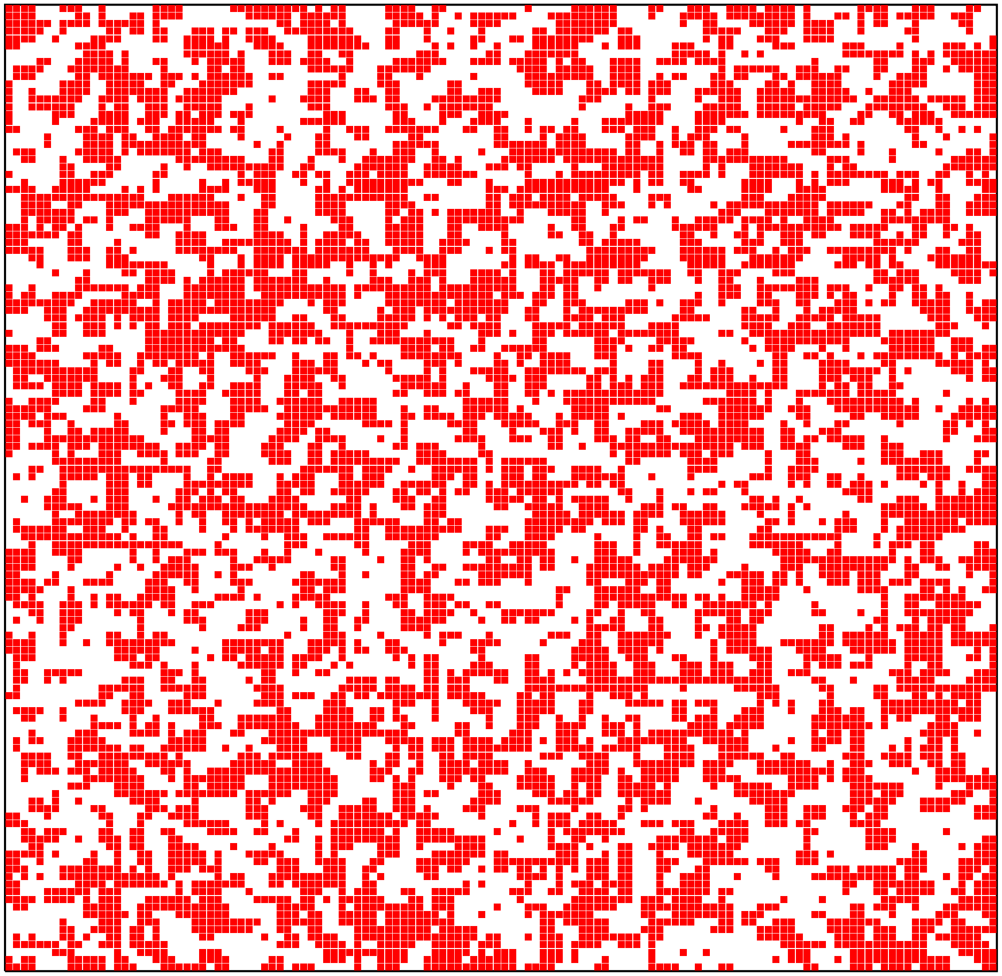
t=0.0

2d square IM quenched to T=0



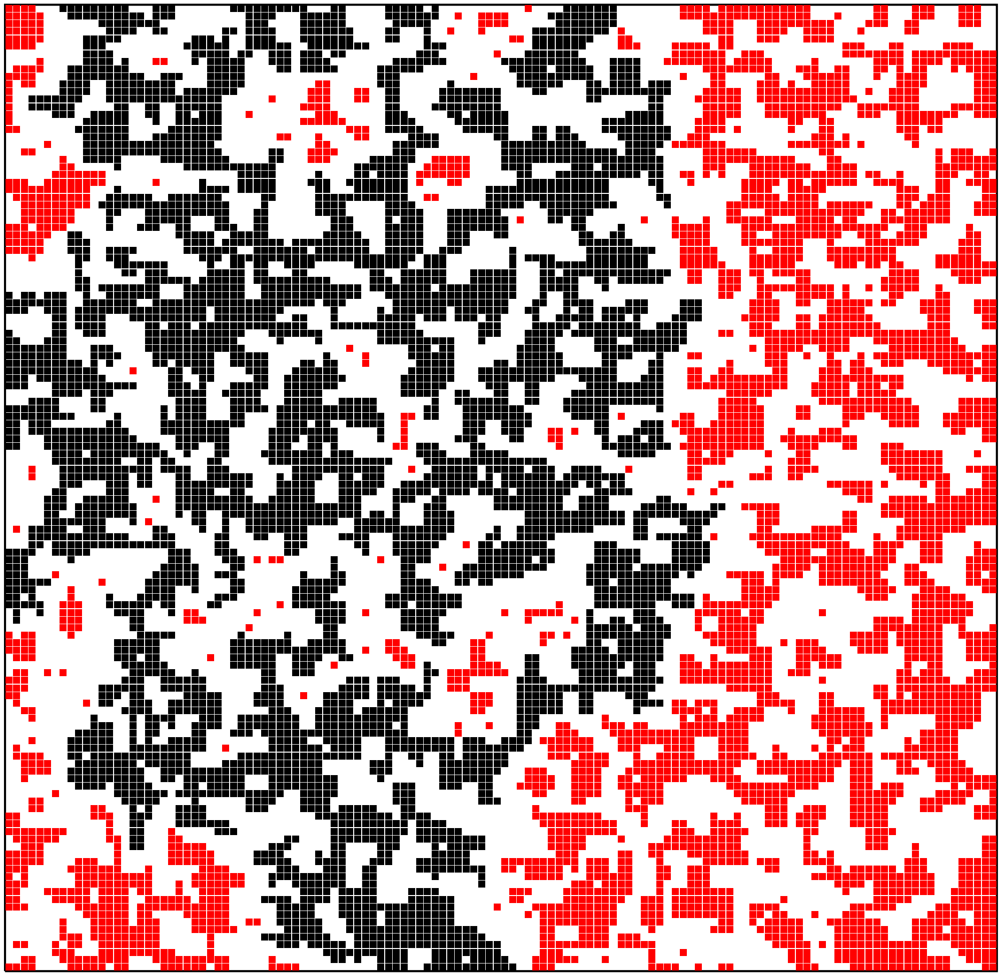
t=0.57533

2d square IM quenched to T=0



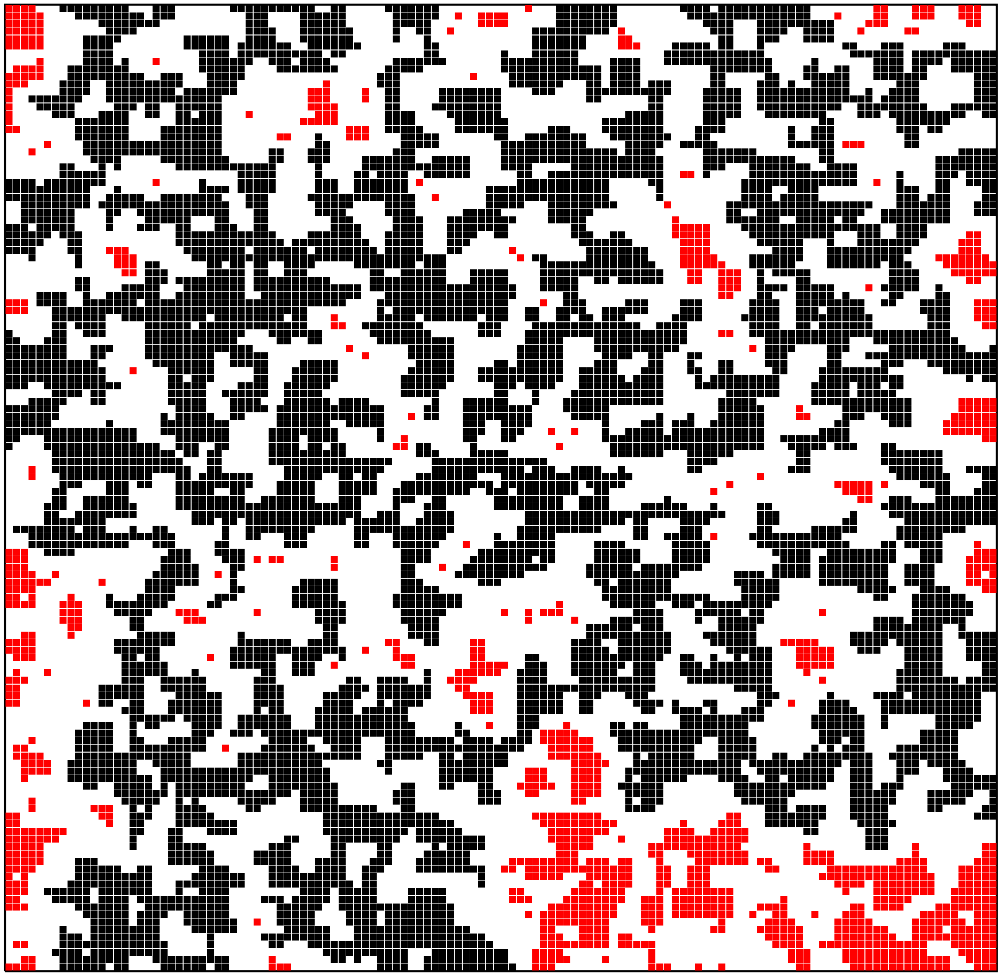
t=0.94844

2d square IM quenched to T=0



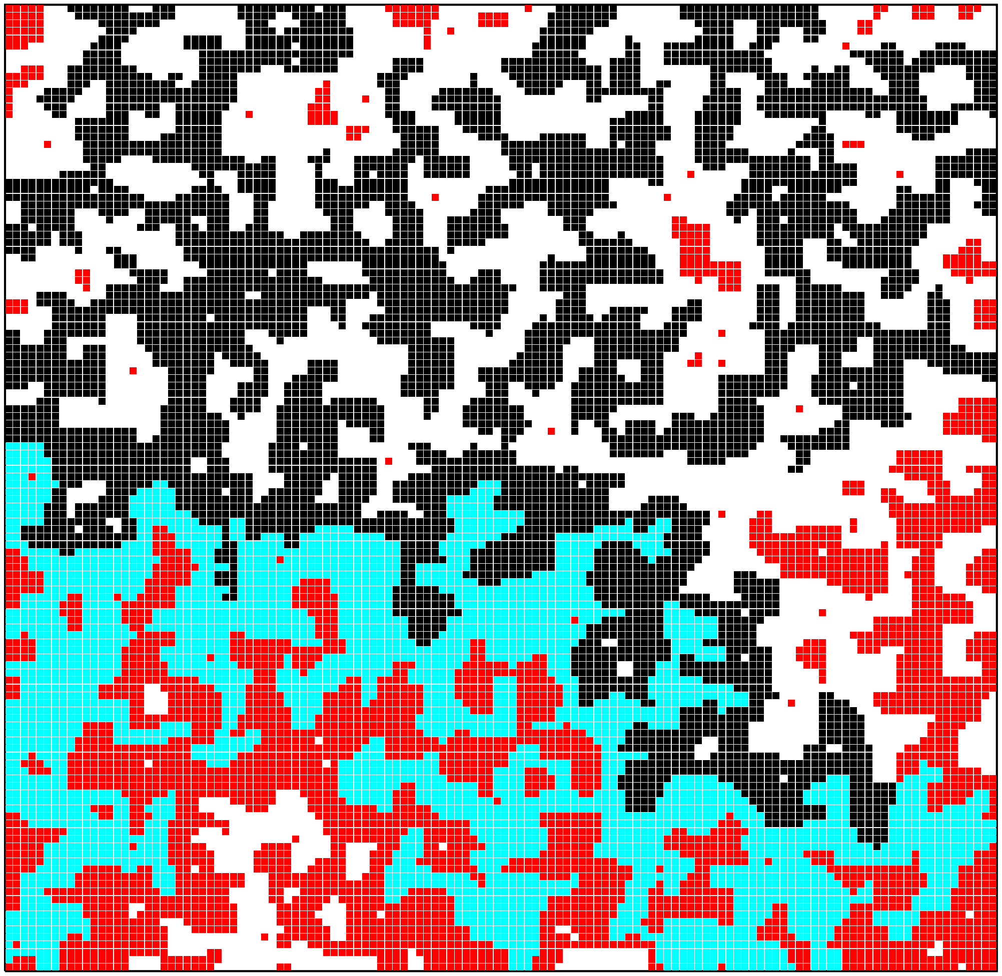
t=2.00847

2d square IM quenched to T=0



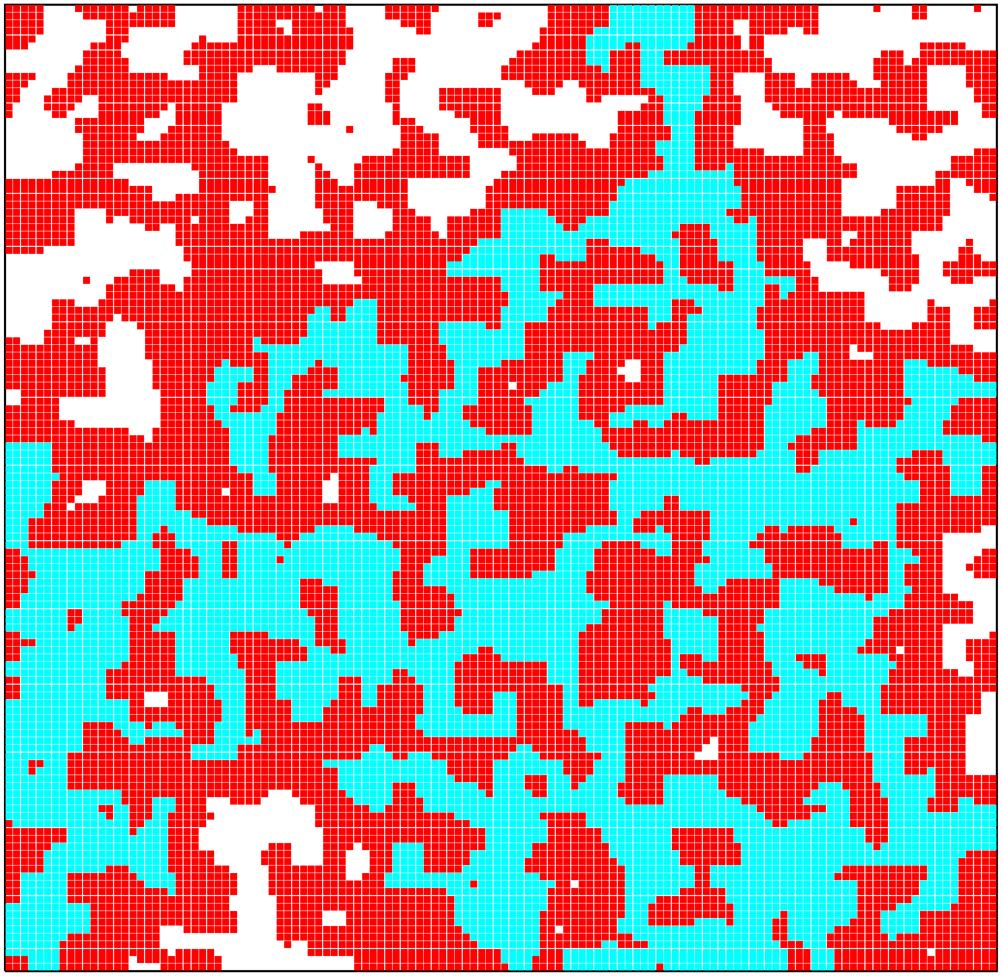
t=2.57898

2d square IM quenched to T=0



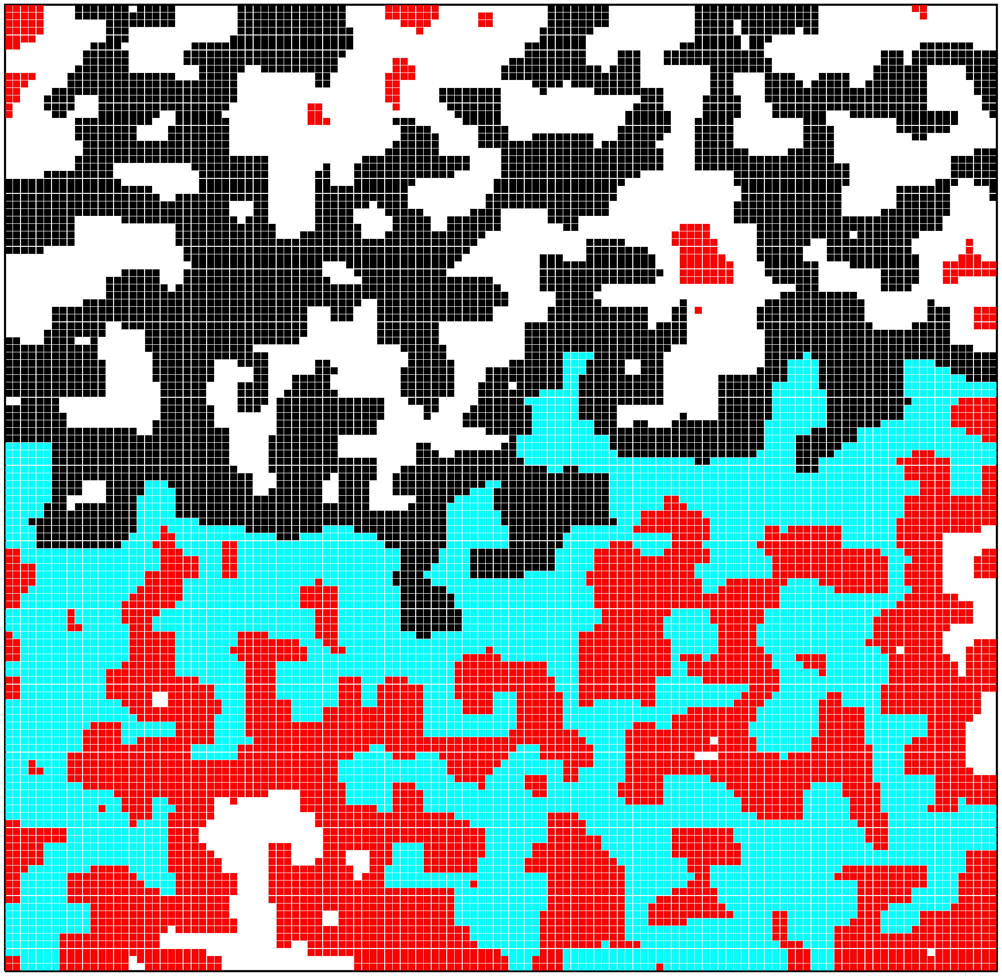
t=3.99211

2d square IM quenched to T=0



t=6.58423

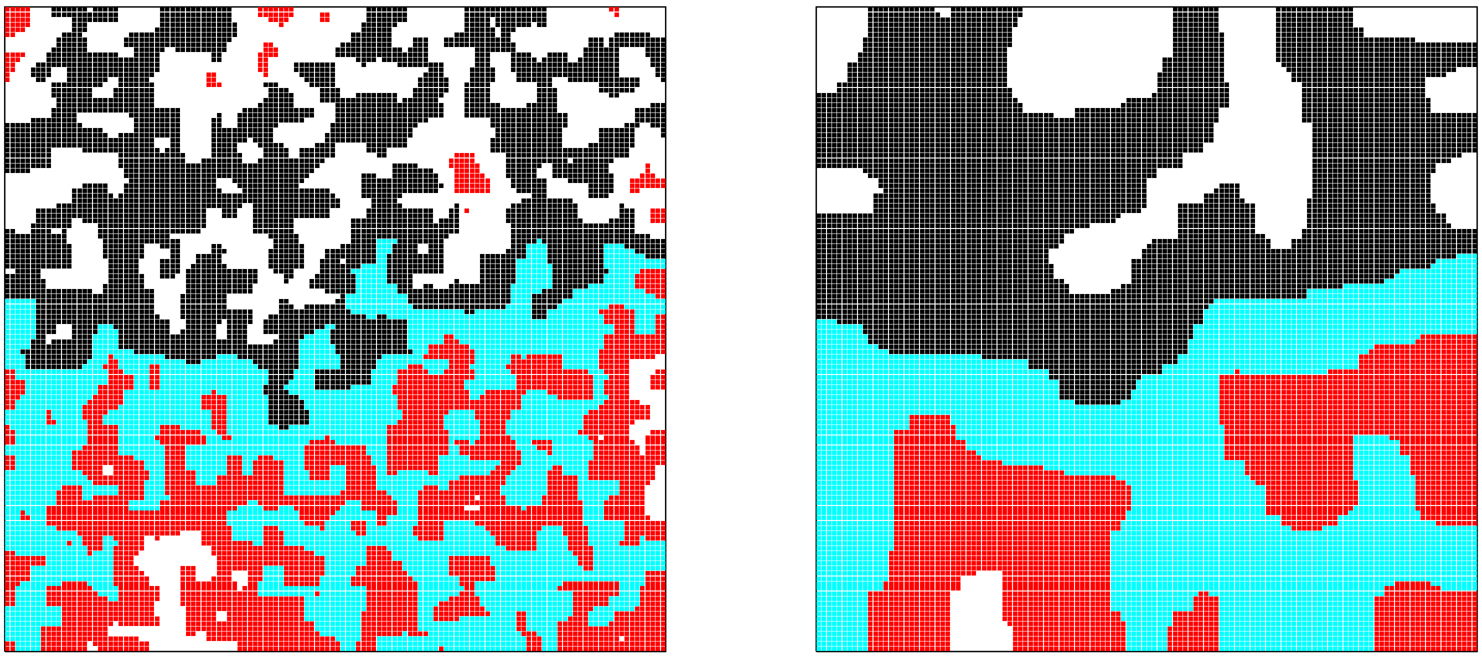
2d square IM quenched to T=0



t=7.46144

2d square IM quenched to T=0

The percolating structure was decided at $t_p \simeq 8$ MCs

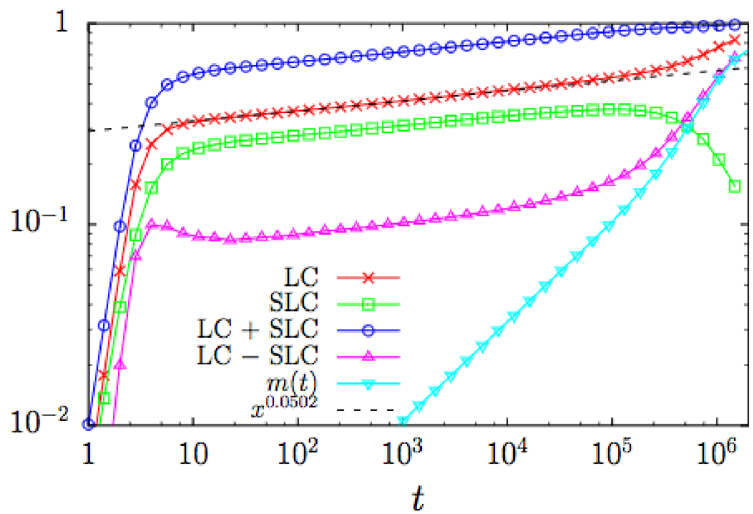


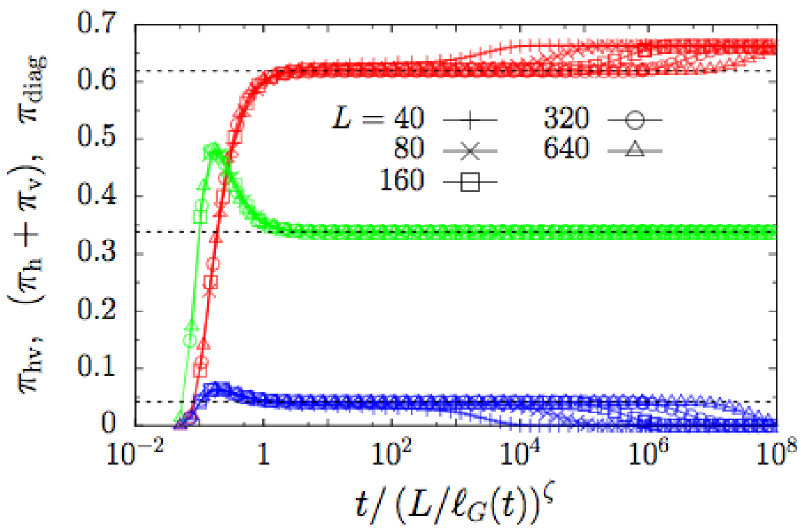
t=7.46144 t=128.0

Arenzon, Bray, LFC & Sicilia 07 Blanchard, Corberi, LFC & Picco 14

2d Ising model quenched to T=0

Evolution of the 2 largest domains

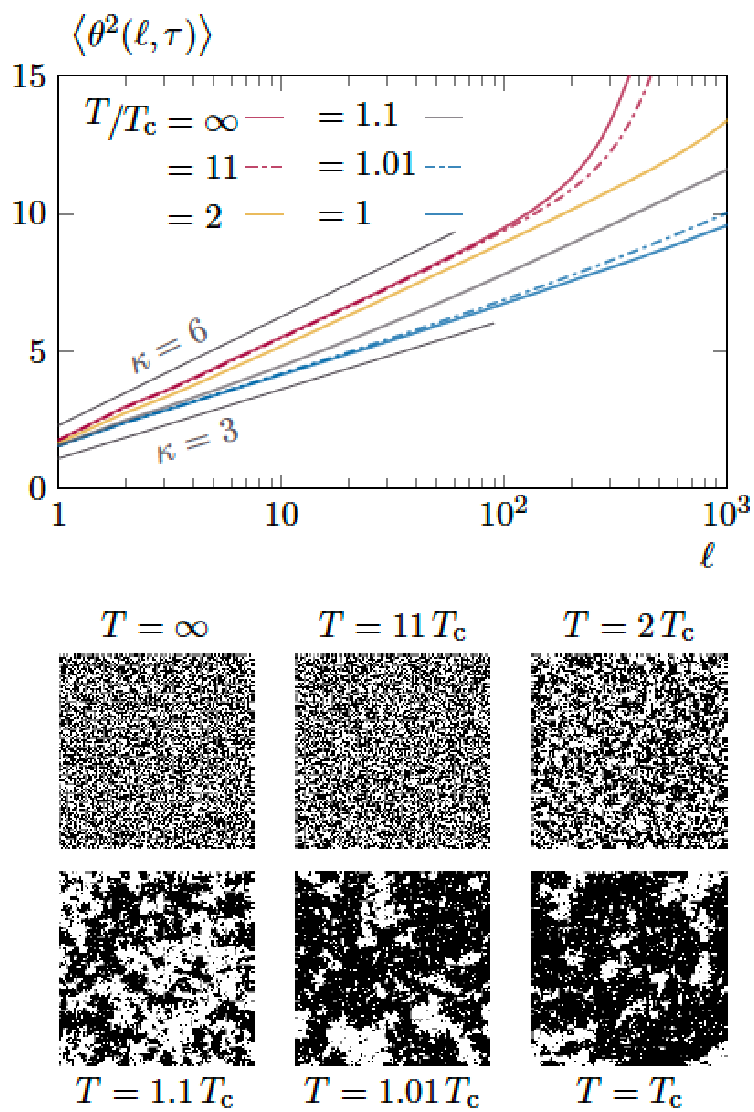




Blanchard, Ricateau, Tartaglia, LFC & Picco

2d IM in equilibrium at $T > T_c$

Geometric properties



Second moment of the winding angle of the longest interface

SLE/CFT arguments

$\kappa = 6$ critical percolation

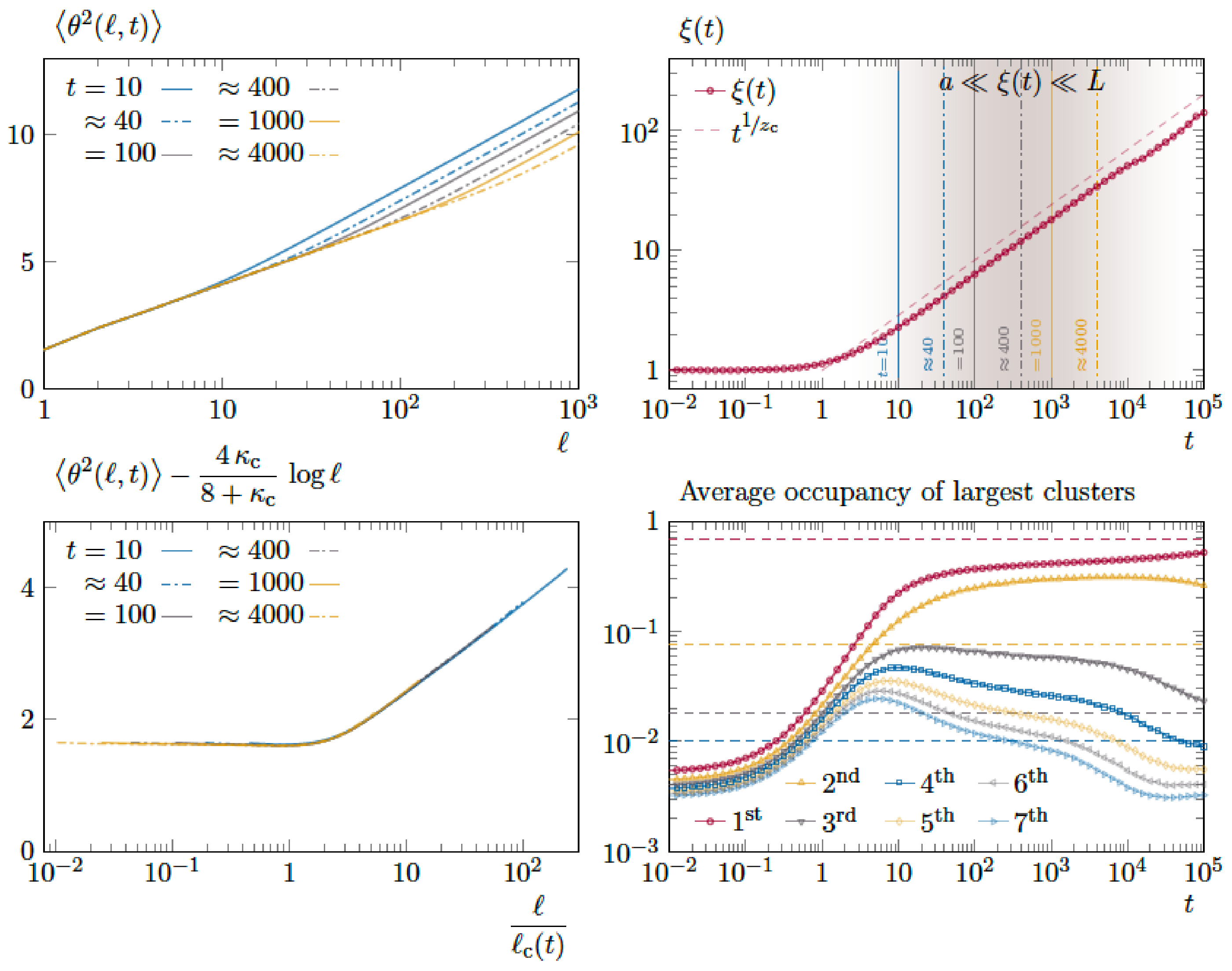
$\kappa = 3$ critical Ising

At short distances ℓ critical percolation properties at high T

Ricateau 17

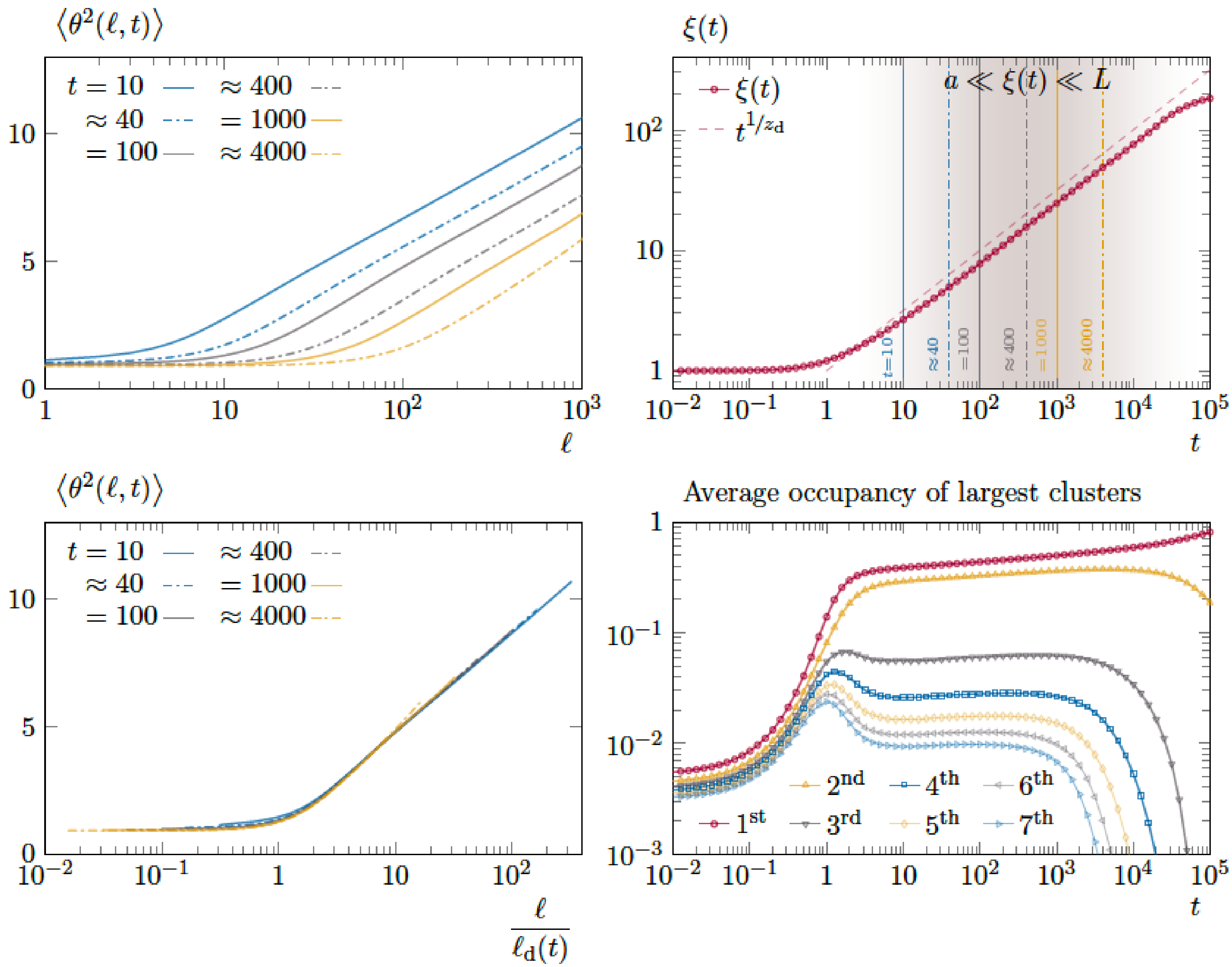
2d square IM quenched to Tc

Geometric properties



2d square IM quenched to T=0

Geometric properties



Complex field & cold atoms

Complex field theory in $3d$

Relativistic bosons; ^{4}He, type II superconductors, cosmology, etc.

$$-c^{-2}\ddot{\psi} + \nabla^2\psi + 2i\mu\dot{\psi} = g(\psi^2 - \rho)\psi$$

c is the velocity of light, ρ and g parameters in (Mexican hat) potential.

Limits

$\mu \to 0$: $$-c^{-2}\ddot{\psi} + \nabla^2\psi = g(|\psi|^2 - \rho)\psi$$ Goldstone

$c \to \infty$: $$2i\mu\dot{\psi} + \nabla^2\psi = g(|\psi|^2 - \rho)\psi$$ Gross-Pitaevskii

models

Complex field theory in $3d$

Relativistic bosons; ^{4}He, type II superconductors, cosmology, etc.

$$-c^{-2}\ddot{\psi} + \nabla^2\psi + 2i\mu\dot{\psi} = g(\psi^2 - \rho)\psi$$

The energy functional

$$E = \int d^3x \, \left(c^{-2}|\dot{\psi}|^2 + |\vec{\nabla}\psi|^2 - g\rho\psi^2 + g\psi^4\right)$$

is conserved under the dynamics.

The energy is minimised by the static configuration $\psi = \sqrt{\rho}\, e^{i\chi}$ with $\chi =$ ct

There are static vortex solutions, e.g. $\psi(\vec{x}) = f(r)\, e^{in\theta}$ with $f(0) = 0$ and $f(r \to \infty) = \sqrt{\rho}$, and $n \in$ Z (thin tubes at the centre of which the field vanishes and the phase turns around).

Tsubota, Kasamatsu & Kobayashi 13, Kobayashi & Nitta 15, etc.

Complex field theory in $3d$

Stochastic noise and dissipation added

$$\boxed{-c^{-2}\ddot{\psi} + \nabla^2\psi + 2i\mu\dot{\psi} - \gamma\dot{\psi} = g(\psi^2 - \rho)\psi - \sqrt{\gamma T}\xi}$$

Langevin-like dynamics

$-\gamma$ viscosity, ξ complex Gaussian white noise in normal form

$\langle \xi_i(\vec{x}, t) \rangle = 0$ and $\langle \xi_i(\vec{x}, t_1)\xi_j(\vec{y}, t_2) \rangle = \delta_{ij}\delta^{(3)}(\vec{x} - \vec{y})\delta(t_1 - t_2)$

Passage to Fokker-Planck formalism allows to show that the dynamics takes the system to

$$\boxed{\lim_{t\to\infty} P(\psi, t) = P_{GB}(\psi) \propto e^{-\beta E}}$$

Kobayashi & LFC 16

Complex field theory in $3d$

Relativistic bosons; ^{4}He, type II superconductors, cosmology, etc.

$$-c^{-2}\ddot{\psi} + \nabla^2\psi + 2i\mu\dot{\psi} - \gamma\dot{\psi} = g(\psi^2 - \rho)\psi - \sqrt{\gamma T}\xi$$

Langevin-like dynamics

$-\gamma$ viscosity, ξ Gaussian white noise in normal form

In the limit $c \to \infty$, the stochastic Gross-Pitaevskii equation

$$(2i\mu - \gamma)\dot{\psi} = -\nabla^2\psi + g(\psi^2 - \rho)\psi + \sqrt{\gamma T}\xi$$

Gardiner et al 00s

$3d$ XY lattice model

Archetypical classical magnetic example

$$H = -J \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j$$

$J > 0$ ferromagnetic coupling constant.

$\langle ij \rangle$ sum over nearest-neighbours on a $3d$ lattice

$\vec{s}_i$ planar spins: two components with constant modulus $\Rightarrow$ angle θ_i.

Second order phase transition with spontaneous symm breaking at $T_c > 0$.

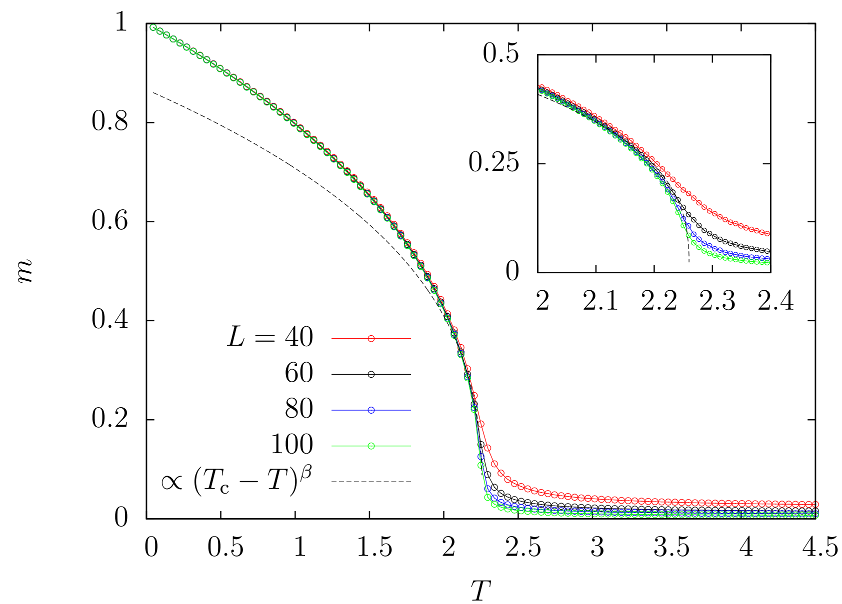
Order parameter: spin-alignment, $\vec{m} \equiv N^{-1} \sum_i \langle \vec{s}_i \rangle$.

No intrinsic spin dynamics, Monte Carlo rules mimic coupling to thermal bath.

Non-conserved order parameter dynamics [$\uparrow\downarrow$ towards $\uparrow\uparrow$] etc. allowed.

Statics

Phase transition and order parameter in the field equation



$L^3\, m = |\sum_{ijk}\langle\psi_{ijk}\rangle|$

critical temperature

$T_c = 2.26$

critical exponent

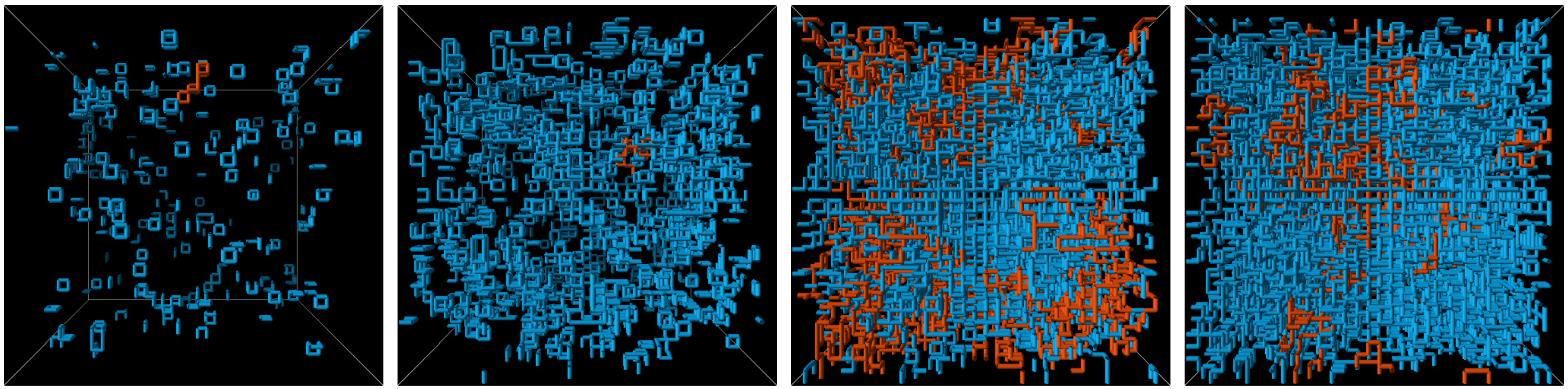
$\beta = 0.347$

Kobayashi & LFC 16

T_c and critical exponents from kurtosis (Binder parameter), susceptibility, specific heat, etc. Values compatible w/results from simulations **Ballesteros et al. 96, Hasenbusch & Török 99** and ϵ expansion **Guida & Zinn-Justin 98**, **Täuber & Diehl 14** for models in the same universality class.

Vortex configurations

In equilibrium



$0.6\,T_c$ $0.8\,T_c$ T_c $1.2\,T_c$

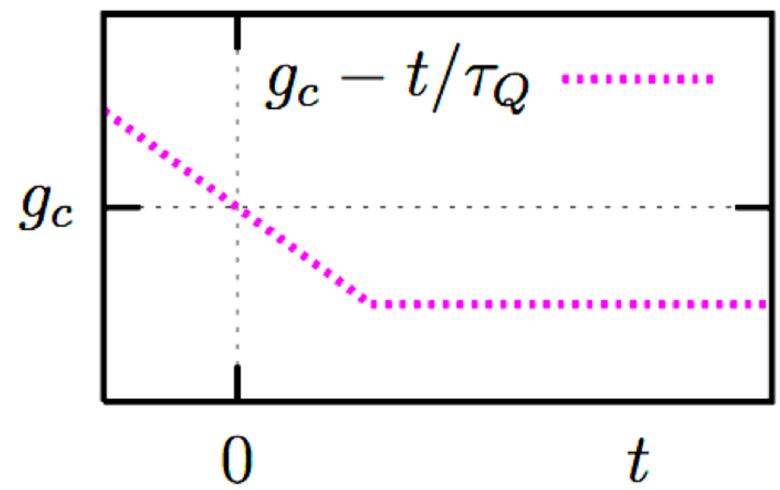
Periodic boundary conditions (torus) implies that the vortex lines are closed, i.e. loops.

Stochastic reconnection rule.

All vortex loops in blue, the longest one in red.

Dynamics after a quench

with g the control parameter



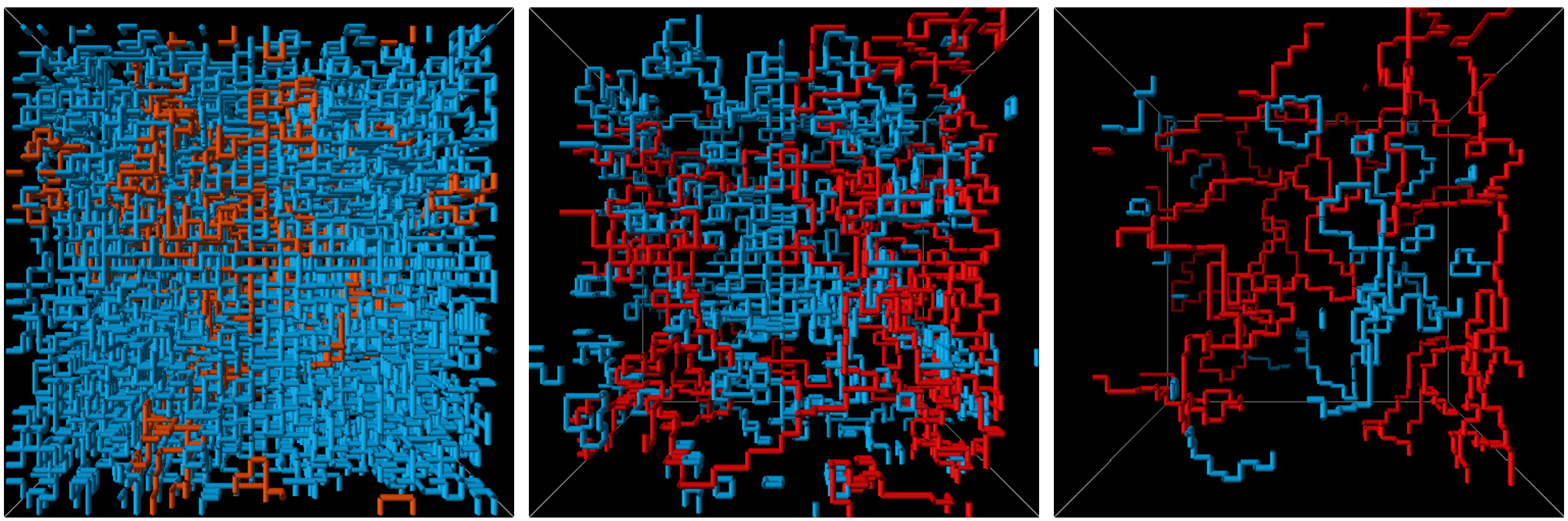
In the picture: annealing with finite rate.

Infinitely fast quench: $T \gg T_c$ for $t < 0$ and $T = 0$ for $t > 0$

Complex field theory in $3d$

Progressive elimination of vortex loops after a quench

$T \gg T_c$ $\qquad\qquad$ $T = 0$



$t = 0$ $\qquad\qquad$ $t = 3$ $\qquad\qquad$ $t = 5$

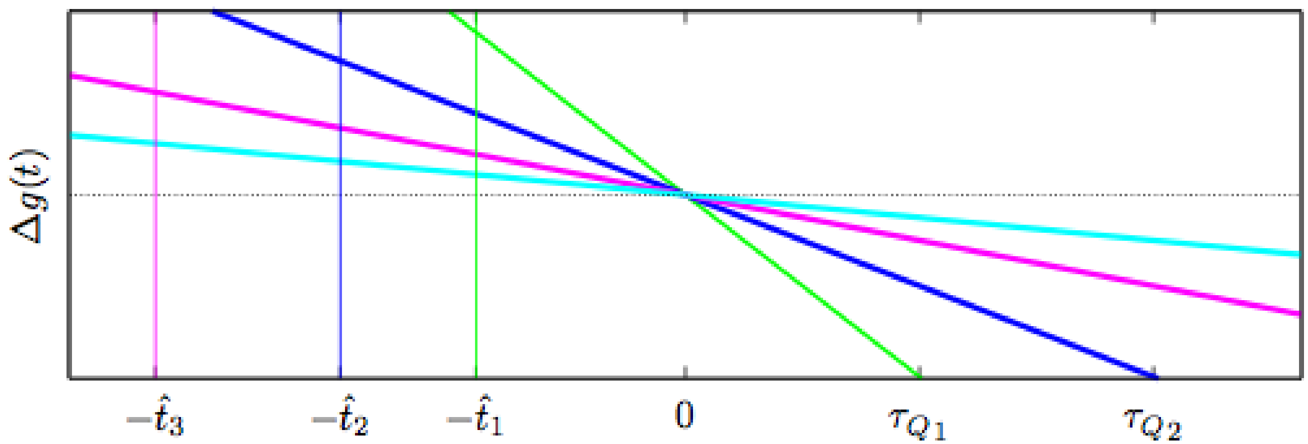
As $\rho_{\text{vortex}} \downarrow$ the reconnection rule loses importance

Kobayashi & LFC 16

Slow cooling & Kibble-Zurek

Finite rate quenching protocol

How is the scaling modified for a very slow quenching rate ?



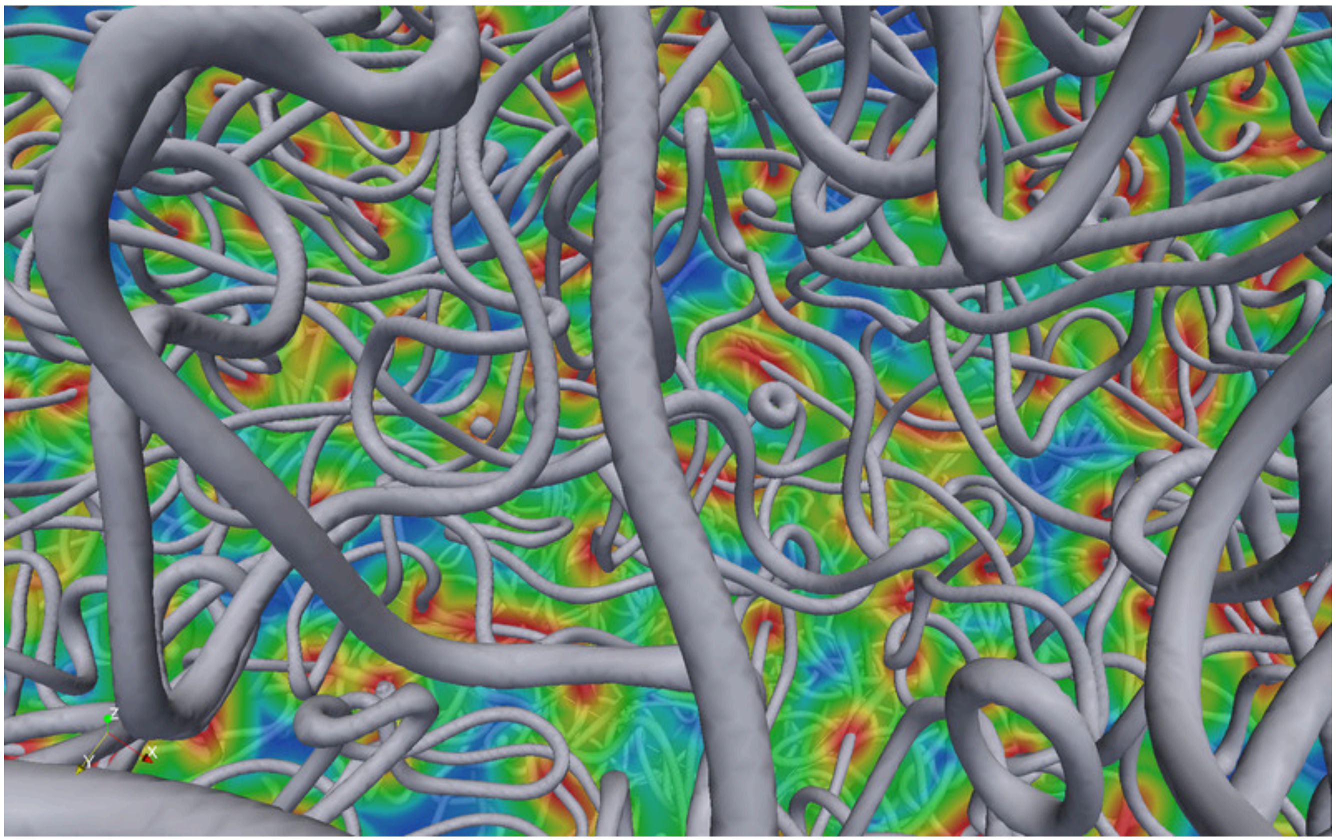
$\Delta g \equiv g(t) - g_c = -t/\tau_Q$ with $\tau_{Q1} < \tau_{Q2} < \tau_{Q3} < \tau_{Q4}$

Standard time parametrization $g(t) = g_c - t/\tau_Q$

Simplicity argument: linear cooling could be thought of as an approximation of any cooling procedure close to g_c.

Theoretical motivation

Network of cosmic strings



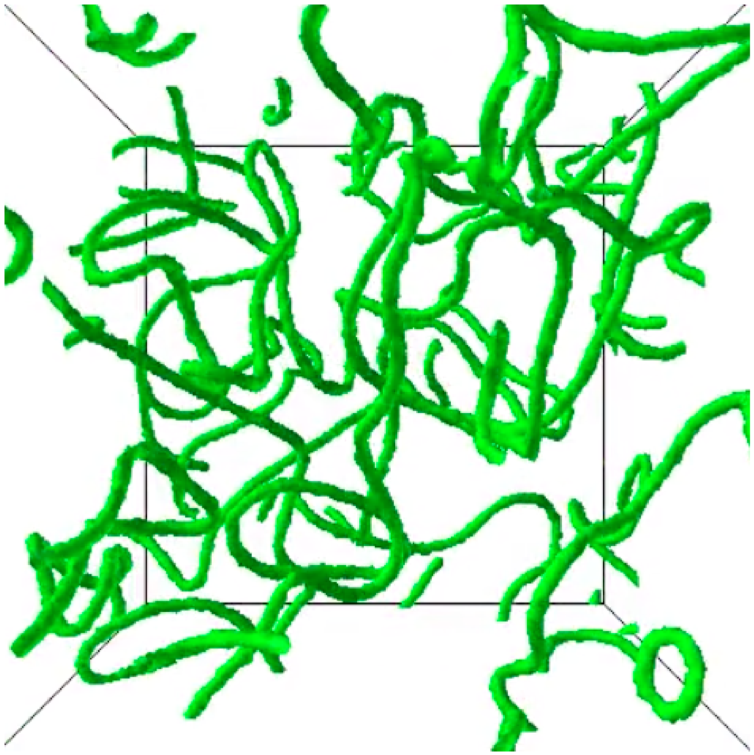
They should affect the Cosmic Microwave Background, double quasars, etc.

Picture from M. Kunz's group (Université de Genève)

Topological defects

instantaneous configurations

Domain walls in the $2d$IM



Vortices in the $3d$ xy model

One can give a precise mathematical definition but the visual one is enough

Density of topological defects

Kibble-Zurek mechanics for 2nd order phase transitions

The three basic assumptions

- Defects are created close to the critical point.
- Their density in the ordered phase is inherited from the value it takes when the system falls out of equilibrium on the symmetric side of the critical point. It is determined by

  Critical scaling above T_c

- The dynamics in the ordered phase is so slow that it can be neglected.

and one claim

- results are universal.

that we critically revisited within 'thermal' phase transitions

Topological defects

after an instantaneous quench : dynamic scaling

$$\Delta n(t) \simeq [\mathcal{R}(t,T)]^{-d} \simeq [\lambda(T(t))]^{-d}\, t^{-d/z_d}$$

Remember the initial ($g \to \infty$) configuration: already there !

Active matter dynamics

$2d$ **dumbbell system**

$|\psi_{6i}|$ **at** $\phi = 0.74$ **and Pe = 10 (coarsening towards co-existence)**