A σ-model for glassy dynamics

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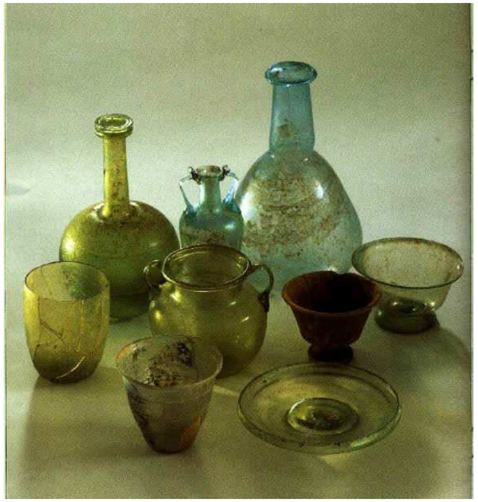
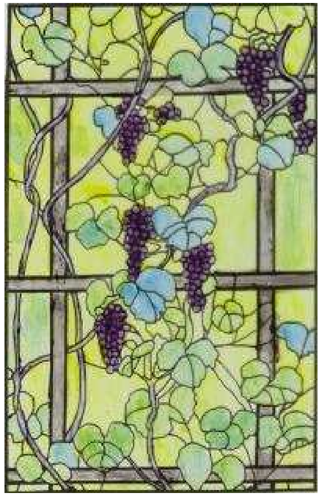
LPS-ENS, 05/04/2006

Plan

- What is the glassy problem ? Overview.
- Some theoretical ideas coming from mean-field theory.
- **Beyond**.

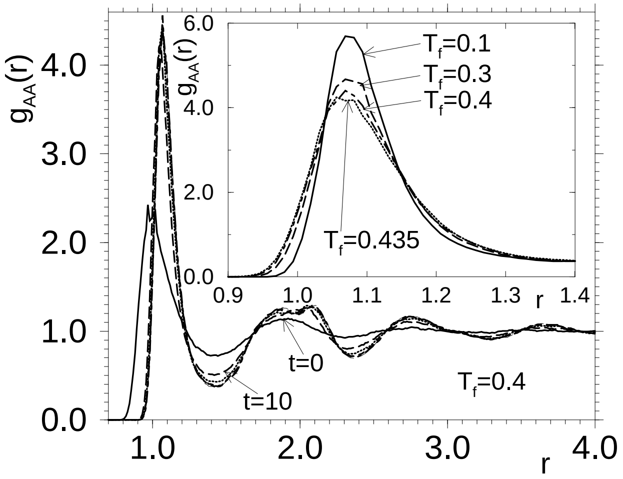


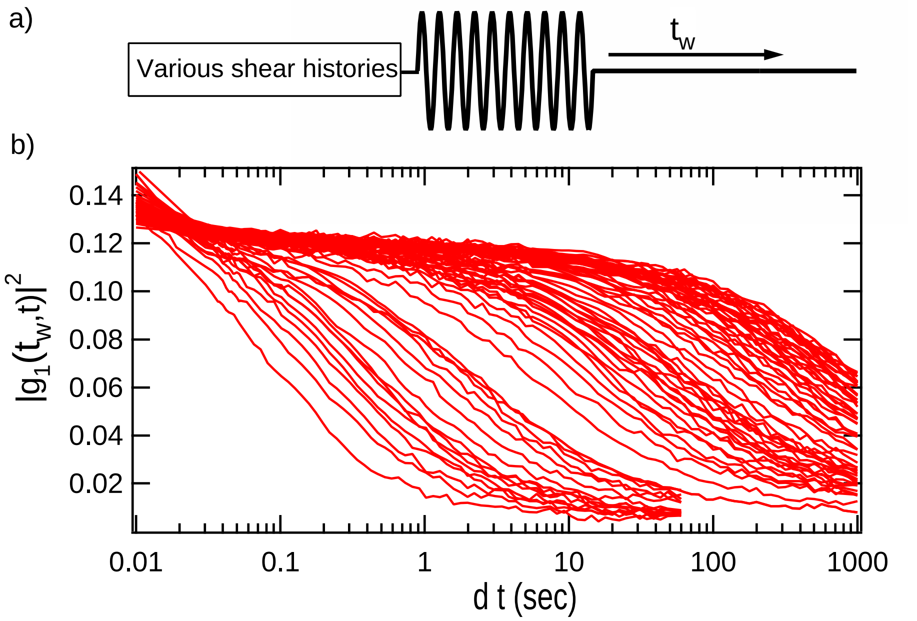


The glassy phenomenon

No obvious structural change

but slowing down !





L-J mixture **J-L Barrat & W. Kob (99)**

Colloidal suspension **B. Viasnoff & F. Lequeux (03)**

$\tau_{micro} \ll \tau_{exp} \ll \tau_{relax}$ that changes by ≈ 10 orders of magnitude !

Time-scale separation & slow non-equilibrium dynamics

Glassy dynamics

Structure factor : nothing special happens at T_g.

One-time quantities decay non-exponentially,

e.g. energy density in a relaxing magnet,

density in a compactifying granular system

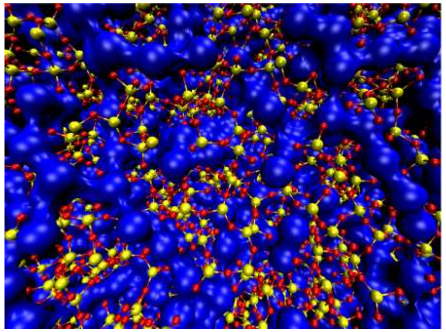
radial distribution function in a particle system

Two-time quantities age, *i.e.* the stationary relaxion is lost

and there is a separation of time-scales, rapid-slow,

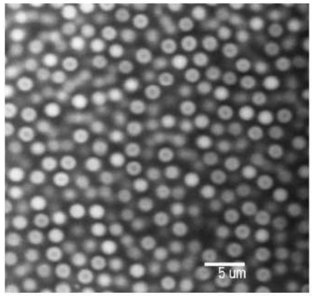
controlled by t_w.

Many systems, many techniques



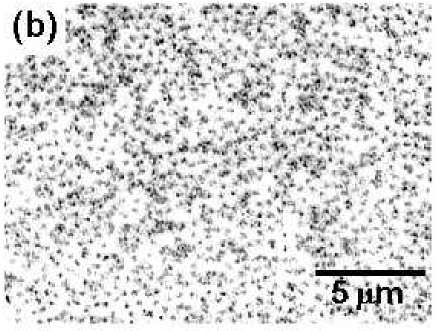
Simulation

Molecular (Sodium Silicate)



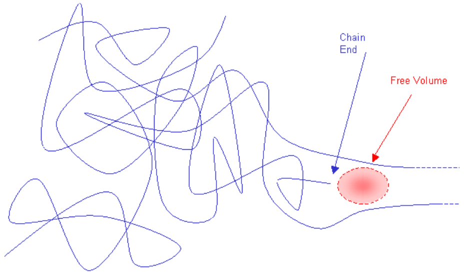
Confocal microscopy

Colloids (e.g. $d \sim 162$nm in water)



Decoration

Vortex ($Bi_2Sr_2CaCu_2O_8$)



Sketch

Polymer melt

Questions

- Can one characterize the global/bulk dynamics ?

(Mean-field/large dimensional models)

- What about the *fluctuations* ? Local/mesoscopic dynamics

Idea : accept the glass without explaining how and why it appears

and describe its dynamics in detail.

(cfr. phonons in solids...)

Focus on two-time quantities.

- Which is the reason for the slowing down ?
- Is there some growing hidden order ?

Modelling

The system is coupled to its environment

$\vec{r}_i$ evolve according to some stochastic rule, *e.g.* Langevin dynamics

$$m\ddot{r}_i^a(t) + \gamma\dot{r}_i^a(t) = -\frac{\delta V(\{\vec{r}_i\})}{r_i^a(t)} + \xi_i^a(t)$$

$$\langle\, \xi_i^a(t)\xi_j^b(t')\,\rangle = 2\gamma k_B T \delta_{ij}\delta^{ab}\delta(t-t')$$

m is a mass, γ the friction coefficient, T is the temperature of the bath and k_B the Boltzmann constant

$V(\{\vec{r}_i\})$ is the potential energy and $-\frac{\delta V(\{\vec{r}_i\})}{\delta r_i^a}$ the deterministic force

Key quantities

Much of the global dynamics can be described with

- the global correlation functions, *e.g.*

$C(t,t_w) = N^{-1}\sum_{i=1}^N \langle s_i(t) s_i(t_w)\rangle$ in spin systems,

$C^s(q;t,t_w) = N^{-1}\sum_{i=1}^N \left\langle e^{i\vec{q}[\vec{r}_i(t)-\vec{r}_i(t_w)]}\right\rangle$ in particle systems.

- their associated linear response functions, *e.g.*

$R(t,t_w) = N^{-1}\sum_{i=1}^N \left.\frac{\delta\langle s_i(t)\rangle}{\delta h_i(t_w)}\right|_{h=0}$ in spin systems.

Solvable models

Large N limit and/or large d limit.

Exact Schwinger-Dyson equations

$$\partial_t C(t,t_w) = \int dt' \; \Sigma(t,t') C(t',t_w) + \int dt' \; D(t,t') R(t_w,t') \; ,$$

$$\partial_t R(t,t_w) = \int dt' \; \Sigma(t,t') R(t',t_w) \; ,$$

where the self-energy and vertex are functions of C and R :

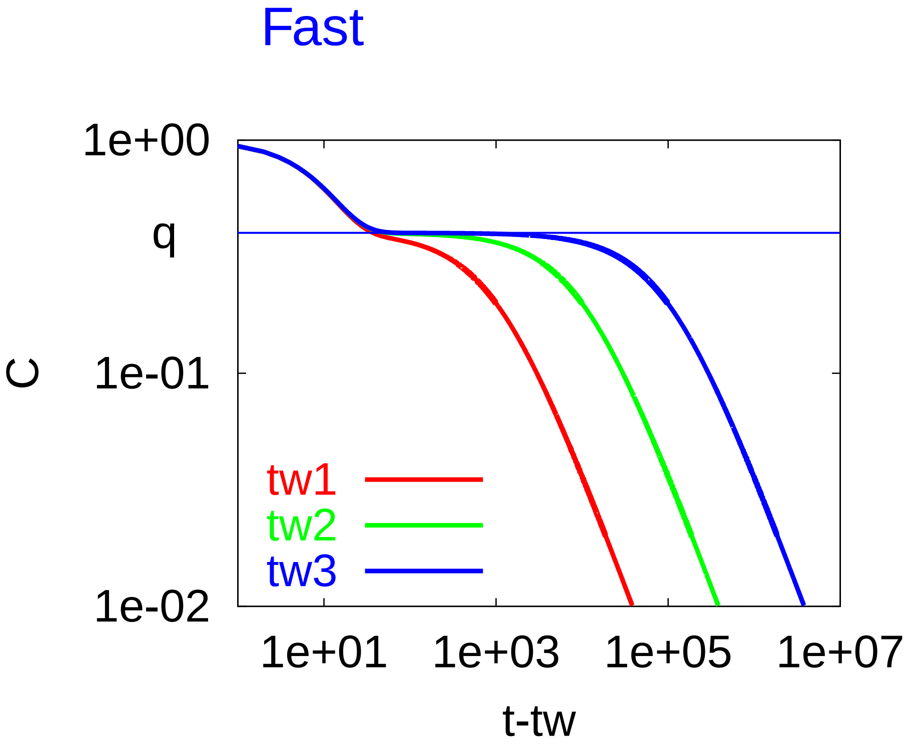
$$D(t,t_w) = D[C(t,t_w)] \; , \qquad \Sigma(t,t_w) = D'[C(t,t_w)]\, R(t,t_w) \; .$$

Solvable numerically and analytically in the long t_w limit.

LFC & J. Kurchan (93)

Separation of time-scales

In the long t_w limit



Slow

$$C^s(t, t_w) \approx f_c\left(\frac{L(t)}{L(t_w)}\right)$$

$$\partial_t C^s(t, t_w) \ll C^s(t, t_w)$$

Eqs. for the slow relaxation $C^s \equiv C < q$:

Approx. asymptotic time-reparametization invariance $t \to h(t)$

Time-reparametrization

Example : the equation $\partial_t R(t,t_w) = \int dt' \; \Sigma(t,t')R(t',t_w)$

- Separation of time-scales (drop $\partial_t R$, approximate the integral) :

$$\mu_\infty R^s(t,t_w) \sim \int dt' \; D'[C^s(t,t')]R^s(t,t') \, R^s(t',t_w) \; . \qquad (1)$$

- The transformation

$$t \to h_t \equiv h(t) \; , \qquad \left\{ \begin{array}{l} C^s(t,t_w) \to C^s(h_t,h_{t_w}) \; , \\ R^s(t,t_w) \to \frac{dh_{t_w}}{dt_w} \, R^s(h_t,h_{t_w}) \; . \end{array} \right.$$

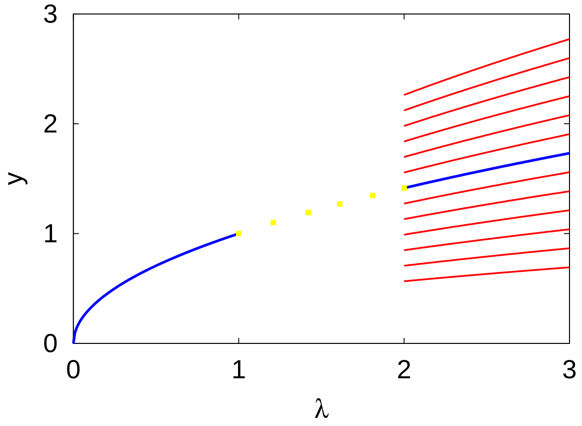
with h_t positive and monotonic leaves eq. (1) **invariant** :

$$\mu_\infty \, R^s(h_t,h_{t_w}) \sim \int dh_{t'} \; D'[C^s(h_t,h_{t'})]R^s(h_t,h_{t'}) \; R^s(h_{t'},h_{t_w}) \; .$$

Time reparametrization invariance

A nuissance

Similar to the matching problem in non-linear diff. eqs.



$$\frac{dy}{d\lambda} = g[y(\lambda)]$$

Many asymptotic solutions.

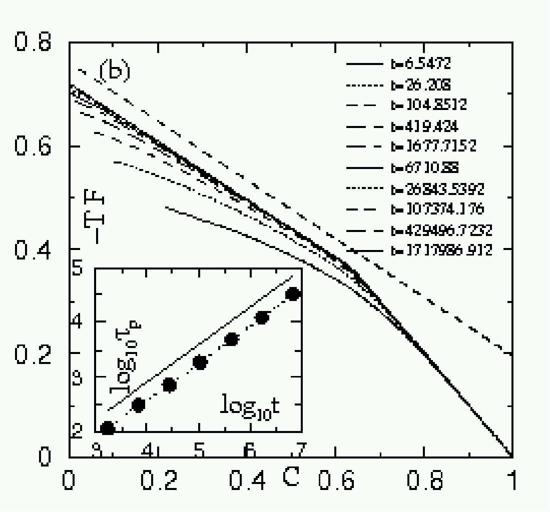
Time reparametrization invariance

One can compute analytically f_c and $\chi^s(C^s)$

$$C^s(t,t_w) \sim f_c\left(\frac{L(t)}{L(t_w)}\right) \;,$$

$$\chi^s(t,t_w) \equiv \int_{t_w}^{t} dt'\; R(t,t') \sim \frac{1-q}{T} + \frac{1}{T_{eff}}\, C^s(t,t_w)$$

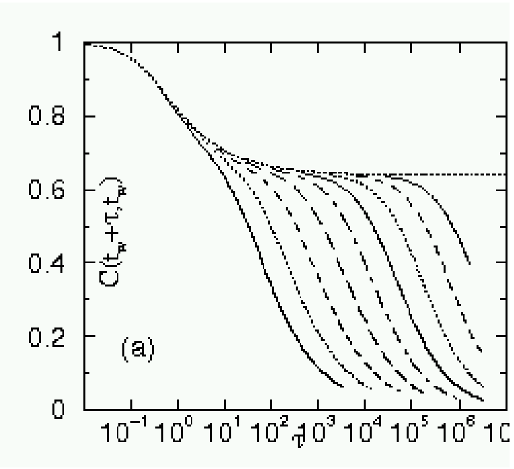
but not the ‘clock’ $L(t)$.



Finite dimensions

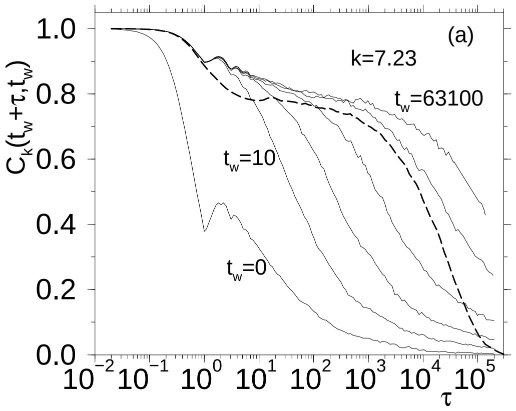
- Slow dynamics : observed
- Separation of time-scales : observed, though less clear-cut.

Num. sol. MF model



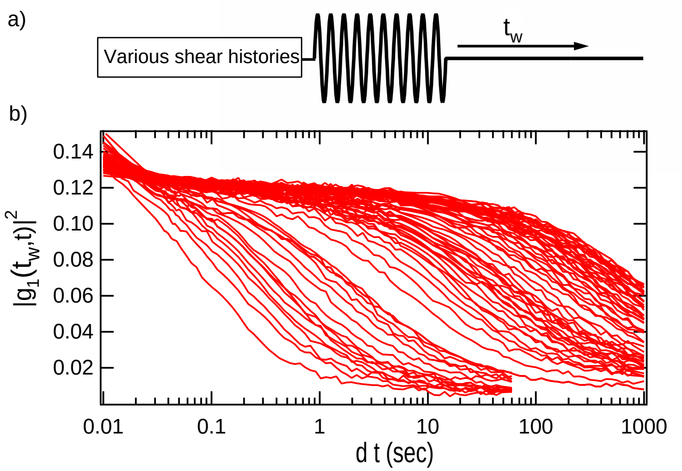
B. Kim & A. Latz (00)

Sim. L-J mixture



J-L Barrat & W. Kob (99)

Exp. colloidal susp.



B. Viasnoff & F. Lequeux (03)

Finite dimensions

Analytically

$$S = S_{slow} + S_{fast} + S_{int}$$

- RG argument based on separation of time-scales allows one to show the approximate asymptotic invariance of S_{slow} under *global* time-reparametrizations

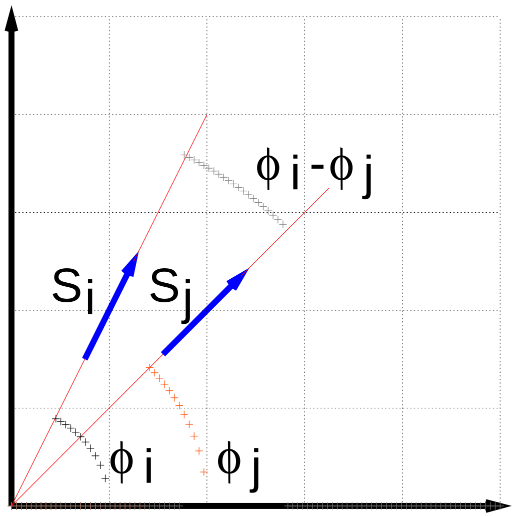
$$t \to h_t \equiv h(t) \;, \qquad \left\{ \begin{array}{l} C_r^s(t,t_w) \to C_r^s(h_t,h_{t_w}) \;, \\ R_r^s(t,t_w) \to \frac{dh_{t_w}}{dt_w}\, R_r^s(h_t,h_{t_w}) \;. \end{array} \right.$$

Symmetry breaking terms become less important as t_w, $t-t_w \to \infty$.

$3d$ Edwards-Anderson spin-glass **Chamon, Kennett, Castillo & LFC (02).**

The Heisenberg ferromagnet

An analogy : a nuissance turned into a model



Landau free-energy

$$F = \int d^d r \left\{ [\nabla \vec{m}(\vec{r})]^2 + \lambda \left[m^2(\vec{r}) - m_0^2 \right]^2 \right\} .$$

Invariant under the global rotation $m^a(\vec{r}) = R^{ab} m^b(\vec{r})$.

(`Global time-reparametrization invariance`)

Statics of the Heisenberg ferro

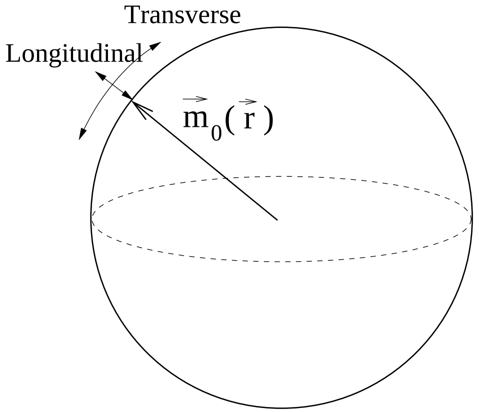
Ground state : $\vec{m}(\vec{r}) = \vec{m}_0$ for all $\vec{r}$.

Fluctuations : $\vec{m}(\vec{r}) = \vec{m}_0 + \delta\vec{m}(\vec{r})$.

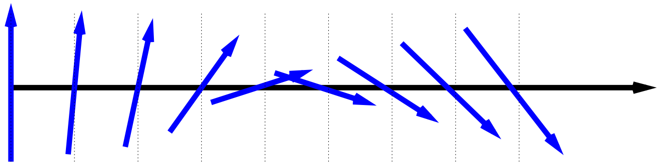
Longitudinal (easy) &

transverse (hard) fluctuations.

Spin-waves



Low energy excitation



(Time-reparametrization waves)

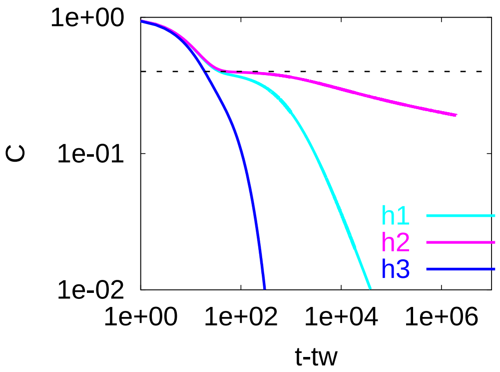
Leading fluctuations

Scaling of the slow part of the global correlation

$$C^s(t,t_w) \approx f_c\left(\frac{L(t)}{L(t_w)}\right) \; .$$

The global time-reparametrization invariance $\Rightarrow \quad C^s_r(t,t_w) \approx f_c\left(\frac{h_r(t)}{h_r(t_w)}\right)$.

Ex. $h_{r_1} = \frac{t}{t_0}$, $h_{r_2} = \ln\left(\frac{t}{t_0}\right)$, $h_{r_3} = e^{\ln^a\left(\frac{t}{t_0}\right)}$ on different regions



Same t_w, slower and faster decays

Castillo, Chamon, LFC, Iguain, Kennett (02,03).

Turn it useful : σ model

Easy fluctuations $t \longrightarrow h_r(t)$.

- Ideally : derive the action $S[h_r(t)]$.

  Doable in quasi-mean-field models, **C. Chamon, LFC, S. Franz, in progress.**

- In practice : propose the action $S[h_r(t)]$;

  derive predictions from $S[h_r(t)]$,

  e.g. $\rho[C_r^s; t, t_w]$; $\rho[R_r^s; t, t_w]$; $\rho[C_r^s, R_r^s; t, t_w]$

  that can be checked numerically & experimentally.

P. Chamon, Charbonneau, LFC, D. Reichman & M. Sellitto (04).

σ-model

Slow decay in terms of $h_r(t) \equiv e^{-\varphi_r(t)}$

$$C_r^s(t,t_w) \approx f_c\left(\frac{h_r(t)}{h_r(t_w)}\right) = f_c\left(e^{-\int_{t_w}^{t} dt' \partial_{t'}\varphi_r(t')}\right)$$

The simplest

(i) global time-reparametrization invariant ;

(ii) local in space ;

(iii) positive definite ($\partial_t h_r(t) > 0 \Rightarrow \partial_t \varphi_r(t) > 0$) ;

(iv) invariant under $\varphi_r(t) \to \varphi_r(t) + \Phi(r)$ as C_r^s effective action is

$$\mathcal{A} = K \int d^d r \int dt \, \frac{[\nabla \partial_t \varphi_r(t)]^2}{\partial_t \varphi_r(t)}$$

σ-model

Using the ‘proper’ time $\tau(t) \equiv \ln L(t)$

with $L(t)$ the “growth” law in the global corr.

& the change of variables $\psi_r^2(\tau) \equiv \partial_\tau \varphi_r(\tau)$

$$
\begin{aligned}
C_r^s(t, t_w) &\approx f_c\left(e^{-\int_{\ln L(t_w)}^{\ln L(t)} d\tau' \psi_r^2(\tau')}\right) \\
\mathcal{A} &= K \int d^d r \int d\tau \, [\nabla \psi_r(\tau)]^2
\end{aligned}
$$

Chamon, Charbonneau, LFC, Reichman & Sellitto (04)

cfr. Bramwell, Holdsworth & Pinton (98) xy-model – spin waves ;

Antal & Rácz (94) Edwards-Wilkinson manifold.

Some consequences

- Temporal scaling of the pdf of local correlations dictated by the global correlation $\rho(C_r; t, t_w) = \rho[C_r; C^s(t, t_w)]$.
- Negatively-skewed, non-Gaussian $\rho(C_r; C^s)$ for $0 < C^s < q$.
- The two-time dependent correlation length $\xi(t, t_w)$,

$$\left[\sum_i C_i^s(t, t_w) C_j^s(t, t_w)\right]_c \approx e^{-|\vec{r}_i - \vec{r}_j|/\xi(t,t_w)} ,$$

should diverge with t and t_w.

- Constant of motion. $\rho[C_r, \chi_r; t, t_w]$ should follow the global FDT rel. :

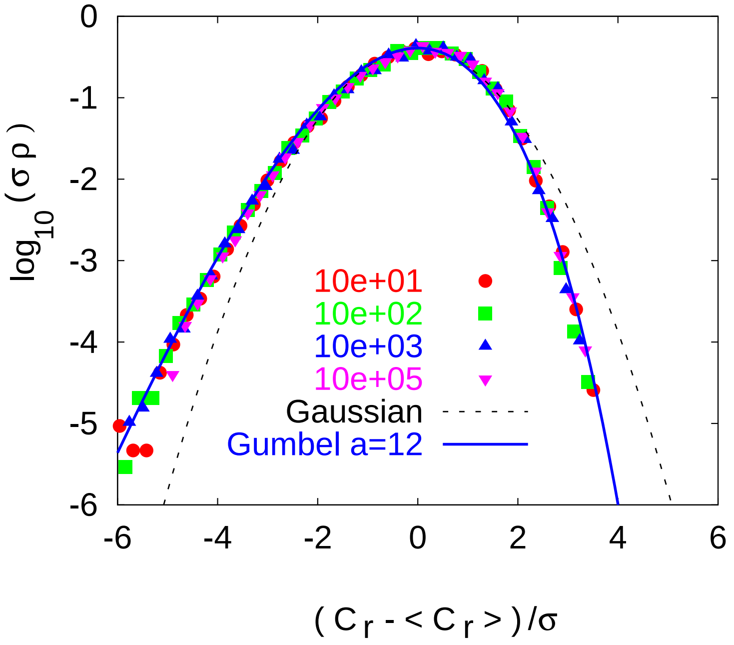
$$\lim_{t_w \to \infty; C(t,t_w) = C} \chi(t, t_w) = \chi(C) .$$

All can be tested with simulations & experiments.

pdf of local correlations

Kinetically constrained model ; four $(t, t_w)/\ C(t, t_w) = 0.8$.

Similar results for the $3d$ spin-glass.



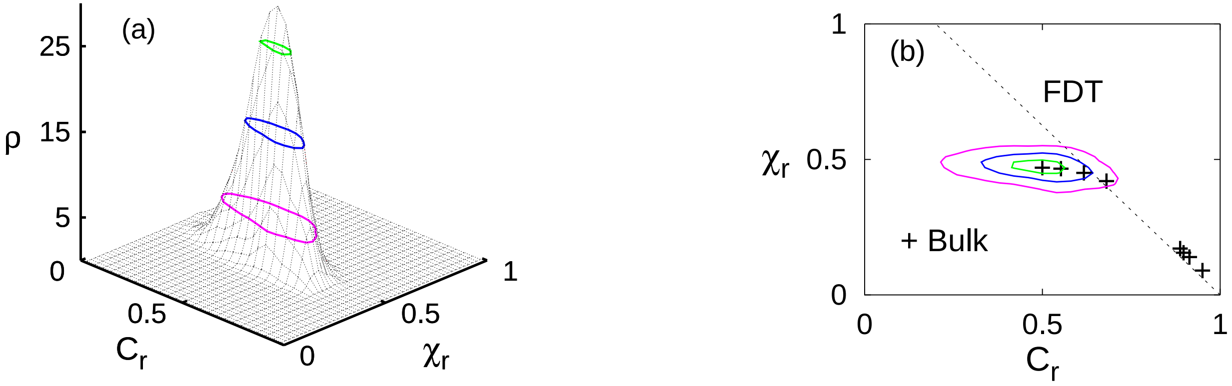
Chamon, Charbonneau, LFC, Reichman & Sellitto (04)

cfr. **E. Bertin (05)**

pdf of correlations & responses

$3d$ Edwards-Anderson spin-glass.

$$C_r(t,t_w) \equiv \frac{1}{V_r}\sum_{i\in V_r} s_i(t)s_i(t_w)\ , \quad \chi_r(t,t_w) \equiv \frac{1}{V_r}\sum_{i\in V_r}\int_{t_w}^{t} dt' \ \left.\frac{\delta s_i(t)}{\delta h_i(t')}\right|_{h=0}$$



+ Bulk : Parametric plot $\chi(t,t_w)$ vs $C(t,t_w)$ for t_w fixed and 7 t $(> t_w)$.

ρ corresponds to the maximum t yielding the smallest C (left-most +).

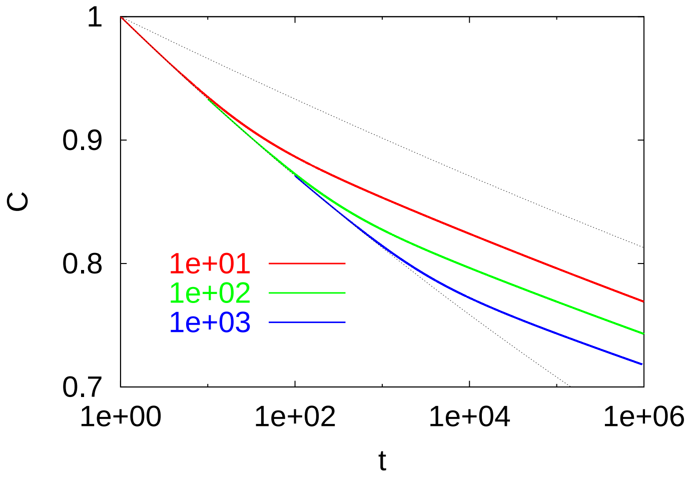
Castillo, Chamon, LFC, Kennett (02).

How general is this ?

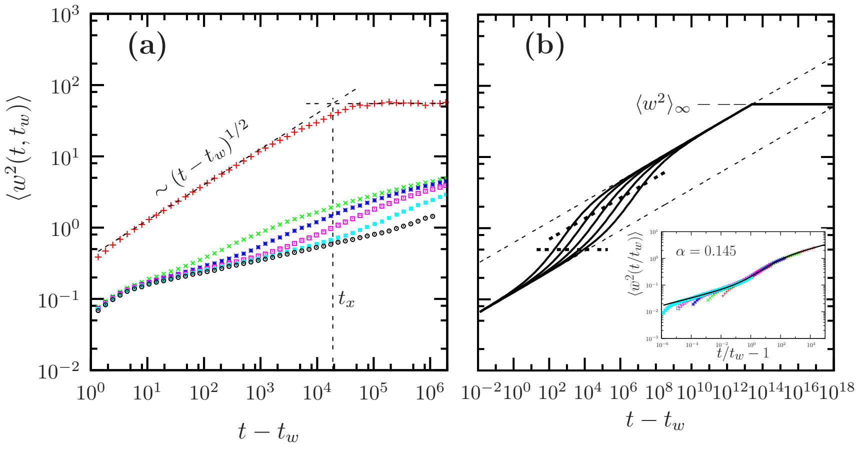
- Critical dynamics

e.g. the $2d$ xy model,

an elastic line in a random environment



Berthier, Holdsworth, Sellitto (03)



Yoshino (96), Bustingorry, Iguain, LFC, Chamon, Domínguez (06)

Multiplicative scaling

$$C(t,t_w) \approx t_w^{-\alpha} f_c\left(\frac{L(t)}{L(t_w)}\right)$$

Take care of $t_w^{-\alpha}$ & saturation !

How general is this ?

- Coarsening – domain growth

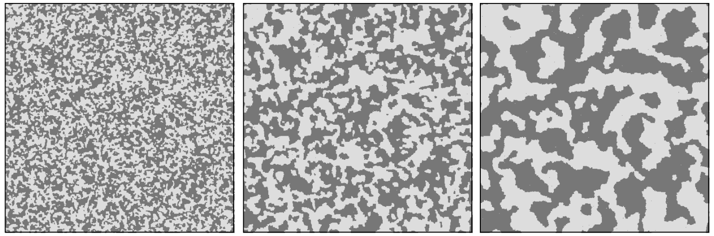
Just scale invariance

e.g. the d-dimensional $O(N)$ model in the large N limit (continuous space limit of the Heisenberg ferro with $N \to \infty$)

$$\dot{\phi}_\alpha(\vec{r},t) = \nabla^2\phi_\alpha(\vec{r},t) + \lambda|N^{-1}\phi^2(\vec{r},t) - 1|\,\phi_\alpha(\vec{r},t) + \vec{\xi}_\alpha(\vec{r},t)$$

Different mechanism, linked to extreme violation of the fluctuation-dissipation equilibrium relation between correlations and responses ($T_{eff} \to \infty$).

Chamon, LFC, Yoshino (06)

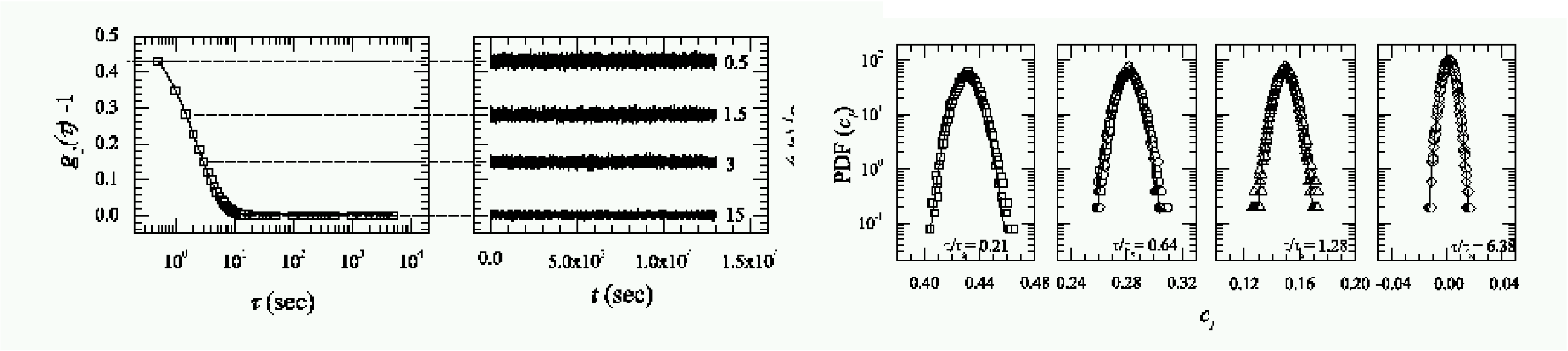


Is it this way for all coarsening systems ?

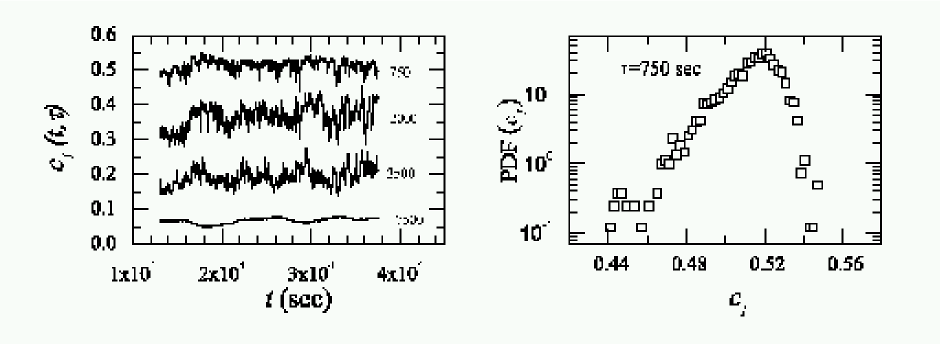
Arenzon, Chamon, LFC, Sicilia, in progress

Experiments

Time fluctuations in Brownian particles



a micellar polycrystal



A. Duri, P. Ballesta, L. Cipelletti, H. Bissig, & V. Trappe (04)

Spatial fluctuations in polymer glasses (cantilevers) **K. Sinnathamby, H. Oukris, N. Israeloff (06)** ;

colloidal suspensions (confocal microscopy) **P. Wang, C. Song, H. Makse et al, in progress**

Summary

Theory for the nonequilibrium dynamics in the glassy phase.

dictated by (the assumption) of

Global time reparametrization invariance

cfr. Spin-waves in Heisenberg ferromagnets.

Predictions for the behaviour of local correlations and responses,

in rather good agreeement with simulations in disordered spin models and kinetically constrained models ; experiments on colloidal systems on their way

Summary

Classification of non-equilibrium systems ?

The theory suggests a strong link between T_{eff} and the fluctuations.

- Structural and spin glasses – aging, $T_{eff} < +\infty$.
- Critical dynamics – interrupted aging, no asymptotic T_{eff}.
- Domain growth – aging in the correlations but ‘no memory’, $T_{eff} \to \infty$

with different properties of the fluctuations.

to be confirmed !