Out of equilibrium dynamics of complex systems

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Plan of Lectures

1. Introduction
2. Coarsening
3. Disorder
4. Active Matter
5. Integrability

First lecture

Plan of the 1st Lecture

Plan

1. Equilibrium *vs.* out of equilibrium classical systems.

2. How can a classical system stay far from equilibrium ?

   From single-particle to many-body

   Diffusion

   Phase-separation & domain growth

   Quenched randomness & glasses

   Driven systems

   Active matter

3. Purposes

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Statistical physics

Advantage

No need to solve the dynamic equations!

Under the *ergodic hypothesis*, after some *equilibration time* $t_{\rm eq}$, *macroscopic observables* can be, on average, obtained with a *static* calculation, as an average over all configurations in phase space weighted with a probability distribution function $\boldsymbol{P}(\{\boldsymbol{p_i}, \boldsymbol{x_i}\})$

$$\langle A \rangle = \int \prod_i d\boldsymbol{p}_i d\boldsymbol{x}_i \, \boldsymbol{P}(\{\boldsymbol{p_i}, \boldsymbol{x_i}\}) \, A(\{\boldsymbol{p}_i, \boldsymbol{x}_i\})$$

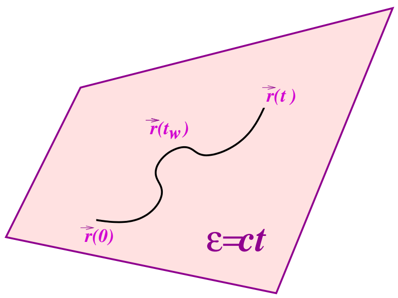
$\langle A \rangle$ should coincide with $\overline{A} \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_{t_{\rm eq}}^{t_{\rm eq}+\tau} dt' \, A(\{\boldsymbol{p}_i(t'), \boldsymbol{x}_i(t')\})$

the *time average* typically measured experimentally

Boltzmann, late XIX

Statistical physics

Ensembles: recipes for $P(\{p_i, x_i\})$ according to circumstances



Isolated system

$$\mathcal{E} = \mathcal{H}(\{\boldsymbol{p}_i, \boldsymbol{x}_i\}) = ct$$

Microcanonical distribution

$$\boxed{P(\{p_i, x_i\}) \propto \delta(\mathcal{H}(\{\boldsymbol{p}_i, \boldsymbol{x}_i\}) - \mathcal{E})}$$

Flat probability density

$$S_{\mathcal{E}} = k_B \ln g(\mathcal{E}) \qquad \beta \equiv \frac{1}{k_B T} = \left.\frac{\partial S_{\mathcal{E}}}{\partial \mathcal{E}}\right|_{\mathcal{E}}$$

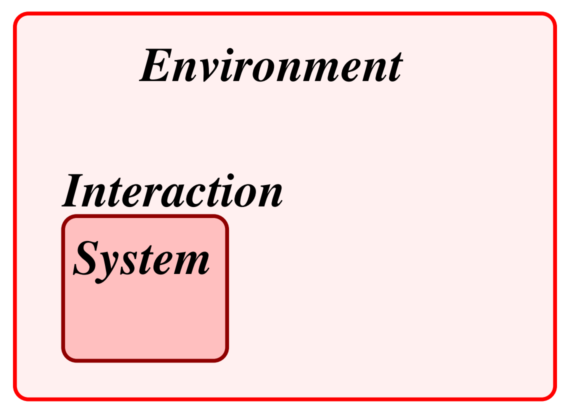
Entropy   Temperature

$$\mathcal{E} = \mathcal{E}_{syst} + \mathcal{E}_{env} + \mathcal{E}_{int}$$

Neglect $\mathcal{E}_{int}$ (short-range interact.)

$$\mathcal{E}_{syst} \ll \mathcal{E}_{env} \qquad \beta = \frac{\partial S_{\mathcal{E}_{env}}}{\partial \mathcal{E}_{env}}$$

$$\boxed{P(\{p_i, x_i\}) \propto e^{-\beta \mathcal{H}(\{\boldsymbol{p}_i, \boldsymbol{x}_i\})}}$$



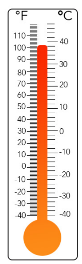
Canonical ensemble

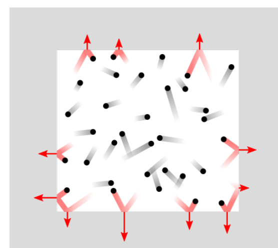
Statistical physics

Accomplishments

- Microscopic definition & derivation of thermodynamic concepts

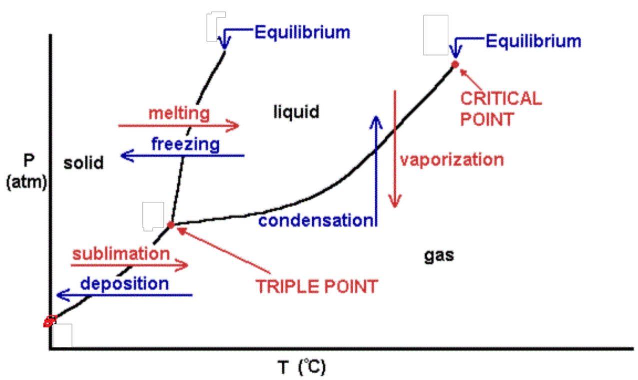
  (temperature, pressure, *etc.*) and laws (equations of state, *etc.*)





$$PV = nRT$$

- Theoretical understanding of collective effects $\Rightarrow$ phase diagrams



Phase transitions : sharp changes in the macroscopic behavior when an external (*e.g.* the temperature of the environment) or an internal (*e.g.* the interaction potential) parameter is changed

- Calculations can be difficult but the theoretical frame is set beyond doubt

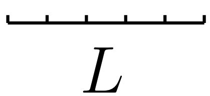
Statistical physics

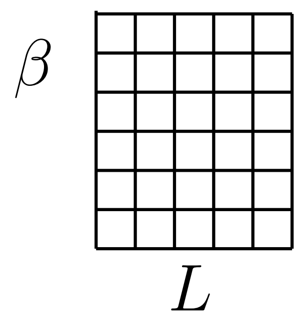
Classical $\Leftrightarrow$ Quantum

Partition function correspondence

Quantum d dimensional $\equiv$ Classical $d+1$ dimensional

$$\mathcal{Z}(\beta) = \text{Tr}\, e^{-\beta \hat{H}} \qquad\qquad \mathcal{Z}(\beta) = \sum_{conf} e^{-\beta \mathcal{H}(conf)}$$





β-periodic imaginary time direction

$$\phi(\boldsymbol{x}) \qquad\qquad \phi(\tau, \boldsymbol{x}) = \phi(\tau + \beta, \boldsymbol{x})$$

Feynman-Hibbs 65, Trotter & Suzuki 76, Matsubara

Quantum Phase transitions, Quantum Monte Carlo methods, *etc.*

Dynamics $\Rightarrow$ Stat Mech

Different cases

- Closed & open systems
- Equilibrium & out of equilibrium
  - Long time scales
  - Forces & energy injection
- Individual & collective effects

General setting

Different cases

- **Closed** & open systems
- Equilibrium & **out of equilibrium**
  - Long time scales
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Isolated systems

Dynamics of a classical isolated system

Foundations of statistical physics.

Question : does the dynamics of a particular system reach a flat distribution over the constant energy surface in phase space ?

Ergodic theory, $\in$ mathematical physics at present.

Dynamics of a (quantum) isolated system :

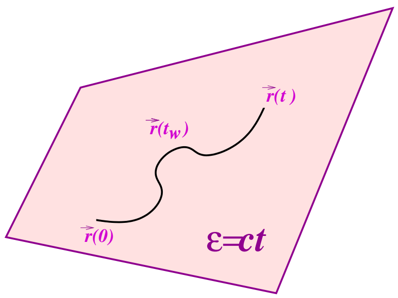
a problem of current interest, recently boosted by cold atom experiments.

Question : after a quench, i.e. a rapid variation of a parameter in the system, are at least some local observables described by canonical thermal ones ? When, how, which ?

5ft lecture

Statistical physics

Ensembles: recipes for $P(\{p_i, x_i\})$ according to circumstances



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Entropy — Temperature

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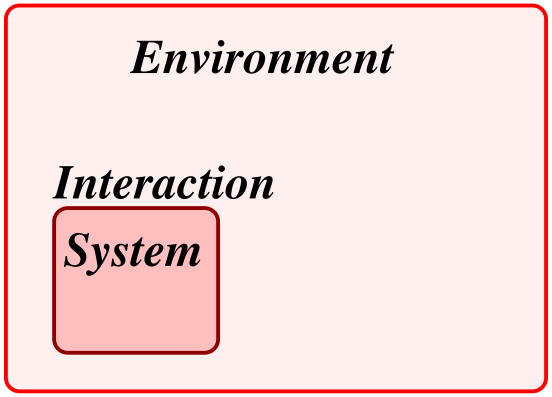


Canonical ensemble

General setting

Different cases

- Closed & **open** systems
- **Equilibrium** & out of equilibrium
  - Long time scales
  - Forces & energy injection
- Individual & **collective effects**

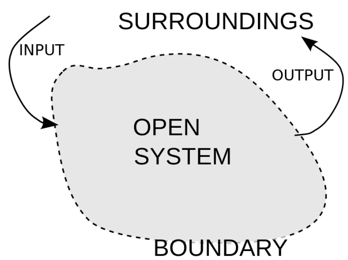
Open systems

Aim

Interest in describing the statics and dynamics of a classical (or quantum) system coupled to a classical (or quantum) environment.

The Hamiltonian of the ensemble is

$$H = H_{syst} + H_{env} + H_{int}$$



The dynamics of all variables are given by Newton (or Heisenberg) rules, depending on the variables being classical (or quantum).

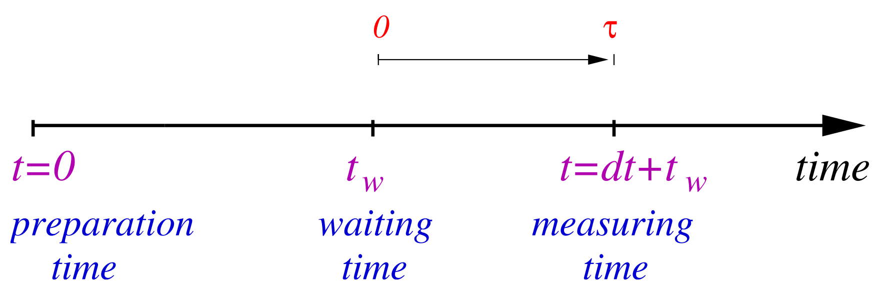
The total energy is conserved, $\mathcal{E} =$ ct but each contribution is not, in particular, $\mathcal{E}_{syst} \neq$ ct, and we'll take $e_0 \ll \mathcal{E}_{syst} \ll \mathcal{E}_{env}$.

In and out of equilibrium

Take a **mechanical point of view** and call $\{\zeta_i\}(t)$ the variables

e.g. the particles' coordinates $\{\boldsymbol{r}_i(t)\}$ and momenta $\{\boldsymbol{p}_i(t)\}$

Choose an initial condition $\{\zeta_i\}(0)$ and let the system evolve.

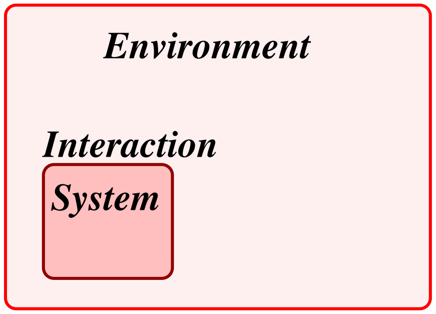


- For $t_w > t_{\rm eq}$: $\{\zeta_i\}(t)$ reach the equilibrium pdf and thermodynamics and statistical mechanics apply (e.g., **temperature** is a well-defined concept).
- For $t_w < t_{\rm eq}$: the system remains out of equilibrium and thermodynamics and (Boltzmann) statistical mechanics **do not** apply.

Dynamics in equilibrium

Conditions

Take an open system coupled to an environment



Necessary :

— The bath should be in equilibrium

same origin of noise and friction.

— Deterministic force

conservative forces only, $\boldsymbol{F} = -\boldsymbol{\nabla} V$.

— Either the initial condition is taken from the equilibrium pdf, or the latter should be reached after an equilibration time $t_{\rm eq}$:

$$P_{\rm eq}(\boldsymbol{v}, \boldsymbol{r}) \propto e^{-\beta(\frac{mv^2}{2} + V(\boldsymbol{r}))}$$

Dynamics in equilibrium

Two properties

- One-time quantities reach their equilibrium values:

$$\langle\, A(\{\boldsymbol{r}\}_\xi)(t)\,\rangle \;\to\; \langle\, A(\{\boldsymbol{r}\})\,\rangle_{\rm eq}$$

[the first average is over realizations of the thermal noise (and initial conditions) and the second average is taken with the equilibrium (Boltzmann) distribution]

- All time-dependent correlations are stationary

$$\langle\, A_1(\{\boldsymbol{r}\}_\xi)(t_1) A_2(\{\boldsymbol{r}\}_\xi)(t_2) \cdots A_n(\{\boldsymbol{r}\}_\xi)(t_n)\,\rangle = \langle\, A_1(\{\boldsymbol{r}\}_\xi)(t_1+\Delta) A_2(\{\boldsymbol{r}\}_\xi)(t_2+\Delta) \cdots A_n(\{\boldsymbol{r}\}_\xi)(t_n+\Delta)\,\rangle$$

for any n and Δ. In particular, $C(t,t_w) = C(t-t_w)$.

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Out of equilibrium

Three possible reasons

- The equilibration time goes beyond the experimentally accessible times in **macroscopic systems** in which $t_{\rm eq}$ grows with the system size,

$$\lim_{N\gg 1} t_{\rm eq}(N) \gg t$$

e.g., **diffusion, critical slowing down, coarsening, glassy physics**

- **Driven systems** Energy injection

$$\boldsymbol{F}_{\rm ext} \neq -\boldsymbol{\nabla} V(\boldsymbol{x})$$

e.g., **active matter**

- **Integrability**

$$I_\mu(\{\boldsymbol{p}_i, \boldsymbol{x}_i\}) = ct, \qquad \mu = 1, \dots, N$$

Too many constants of motion inhibit equilibration to the Gibbs ensembles.

e.g., **1d bosonic gases**

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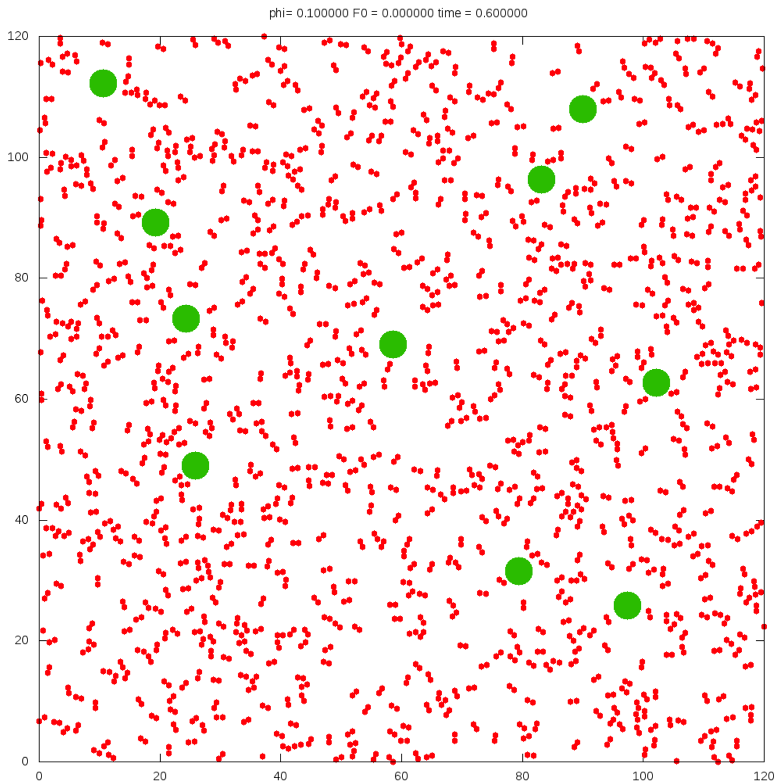
Microscopic system

Brownian motion : diffusion



First example of dynamics of an *open system*

The system : the Brownian particle

The bath : the liquid

Interaction : collisional or potential

Canonical setting

A few Brownian particles or tracers ● embedded in a liquid.

Late XIX, early XX (Brown, Einstein, Langevin)

Langevin approach

Stochastic Markov dynamics

From Newton's equation $\boldsymbol{F} = m\boldsymbol{a} = m\dot{\boldsymbol{v}}$ and $\boldsymbol{v} = \dot{\boldsymbol{x}}$

$$m\dot{v}_a = -\gamma_0 v_a + \xi_a$$

with $a = 1, \ldots, d$ (the dimension of space), m the particle mass,
γ_0 the friction coefficient,
and $\vec{\xi}$ the time-dependent thermal noise with Gaussian statistics,
zero average $\langle \xi_a(t) \rangle = 0$ at all times t,
and delta-correlations $\langle \xi_a(t)\xi_b(t') \rangle = 2\,\gamma_0 k_B T\,\delta_{ab}\,\delta(t-t')$.

Dissipation — for $\gamma_0 > 0$ the averaged energy is not conserved,
$2\langle \mathcal{E}_{syst}(t) \rangle = m\langle v^2(t) \rangle \neq 0$.

Brownian motion

Normal diffusion

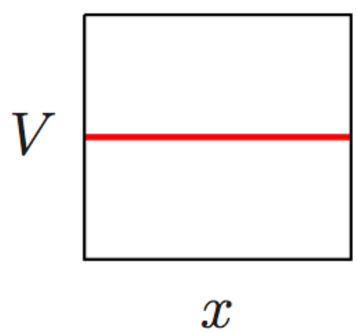
For simplicity, take a one dimensional system, $d = 1$.

The relation between friction coefficient γ_0 and amplitude of the noise correlation $2\gamma_0 k_B T$ ensures equipartition for the velocity variable

$$m\langle v^2(t)\rangle \to k_B T \quad \text{for} \quad t \gg t_r^v \equiv \frac{m}{\gamma_0}$$

Langevin 1908

But the position variable x diffuses since $e^{-\beta V}$ is not normalizable.



$$\langle x^2(t)\rangle \to 2D\,t$$ ($t \gg t_r^v = m/\gamma_0$)

$$D = k_B T/\gamma_0$$ diffusion constant.

The particle is out of equilibrium !

Brownian motion

Normal diffusion

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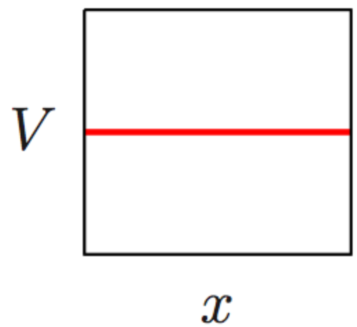


$$\boxed{\langle x^2(t)\rangle \to 2D\, t} \quad (t \gg t_r^v = m/\gamma_0)$$

$$\boxed{D = k_B T/\gamma_0}$$ diffusion constant.

Coexistence of equilibrium (v) and out of equilibrium (x) variables

Macroscopic systems

Discussion of several macroscopic systems with slow dynamics due to

$$\boxed{\lim_{N\gg 1} t_{\rm eq}(N) \gg t}$$

Examples :

Ordering processes **2nd Lecture**

Domain growth, phase separation

Systems with frustrated interactions

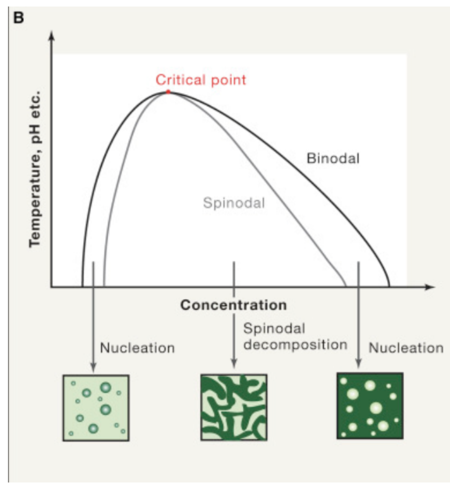
Spin ices

Systems with quenched disorder **3rd Lecture**

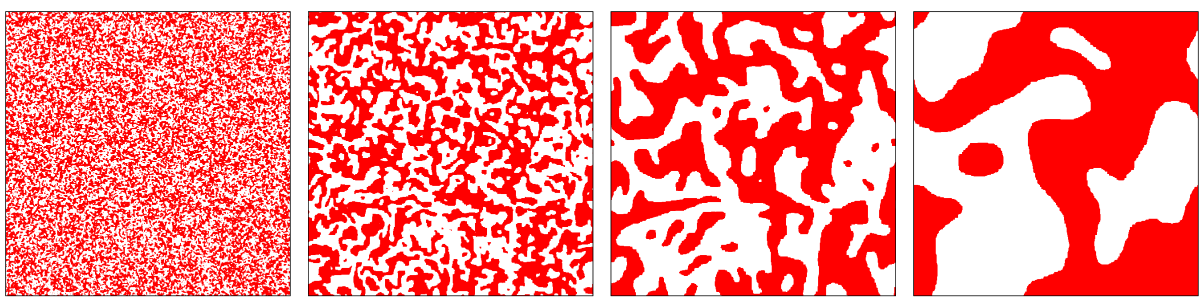
Random ferromagnets, spin-glasses

Phase separation

Quench below the binodal : remnant interfaces



$$t_1 \quad < \quad t_2 \quad < \quad t_3 \quad < \quad t_4 \quad < \quad \dots$$

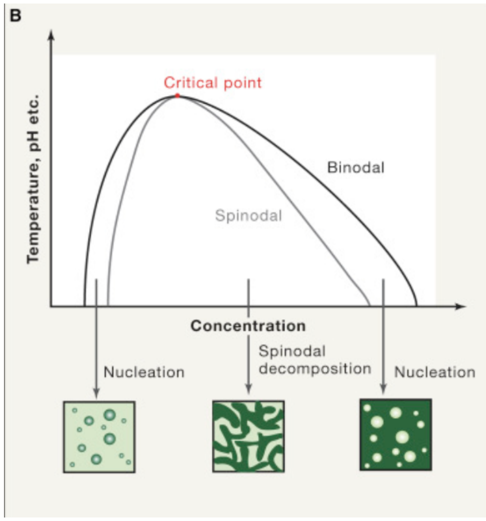


Coarsening process with growing length $\mathcal{R}(t) \simeq t^{1/z} \implies \boxed{t_{\mathrm{eq}} \sim L^z}$

Equilibration time diverges with the system size

Phase separation

Quench below the binodal: universality



Microscopic details are irrelevant

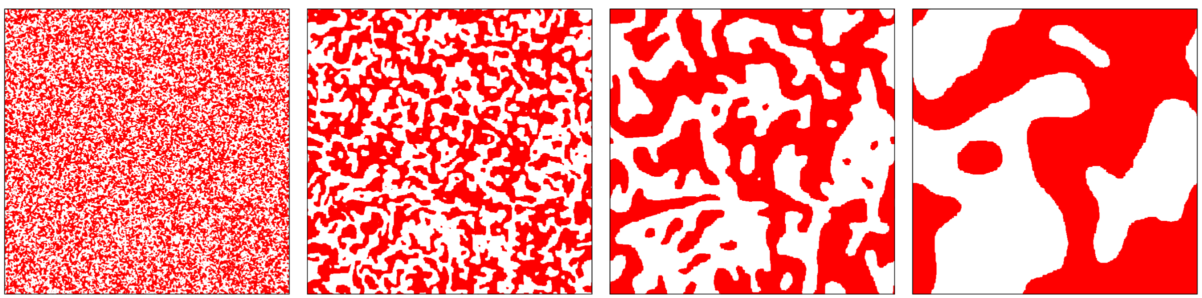
but conservation laws and

dimension of order parameter fix the

$\Longrightarrow$

Dynamic universality class

$t_1 \quad < \quad t_2 \quad < \quad t_3 \quad < \quad t_4 \quad < \quad \dots$



Coarsening process classified according to $\mathcal{R}(t) \simeq t^{1/z}$

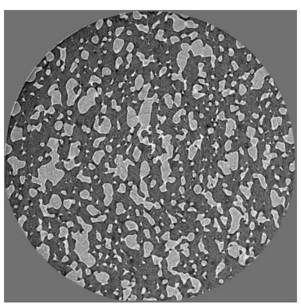
Phase ordering kinetics

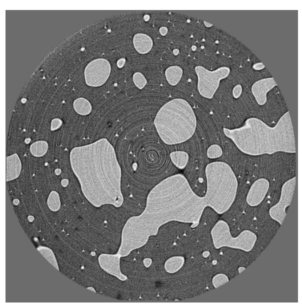
Are these quench dynamics fast processes ? Can we simply forget what happens during the transient, $t_{\rm eq}$, and focus on the subsequent *equilibrium* behaviour ?

No !

It turns out that this is a very slow regime. Its duration grows with the size of the system and it diverges in the thermodynamic limit $N \to \infty$.

We understand the mechanisms for relaxation : interface local curvature driven dynamics and matter diffusion.





The domains get rounder

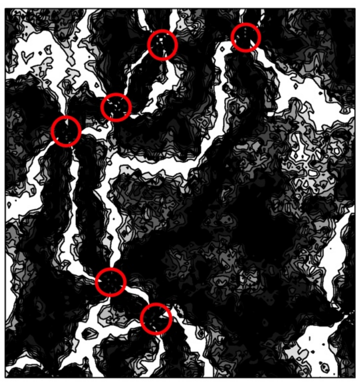
The regions get darker and lighter

Topological phase transitions

Vortices in the $2d$ XY model - O(2) field theory

$$\mathcal{H} = -\frac{J}{2} \sum_{\langle ij \rangle} \boldsymbol{s}_i \cdot \boldsymbol{s}_j \quad \Longrightarrow \int d^2x \, \left[\frac{1}{2} (\boldsymbol{\nabla} \boldsymbol{\phi}(\boldsymbol{x}))^2 - \frac{r}{2} \phi^2(\boldsymbol{x}) + \frac{\lambda}{4} \phi^4(\boldsymbol{x}) \right]$$

Unbinding of vortex pairs $\rho_v^{\text{free}}(T > T_{KT}) > 0$ **Kosterlitz & Thouless 70s**



After a quench to $T < T_{KT}$

Free vortex annihilation

Schlieren pattern

gray scale

$\sin^2(2\boldsymbol{s}_i \cdot \hat{e}_x)$

Jelić & LFC 12

Growing length scale $\mathcal{R}(t) \simeq (t/\ln t)^{1/z}$ & free vortex density $\rho_v^{\text{free}}(t) \sim \mathcal{R}^{-2}(t)$

$$\Longrightarrow \boxed{t_{\text{eq}} \sim L^z \ln L}$$

In boson gases, polaritons, *etc.* **Blakie, Capusotto, Davis, Proukakis, Symanska, ...** numerics & **Beugnon-Dalibard, ... Popovic et al., ...** experiments. Last 10 years

Quenched disorder

Quenched variables are frozen during time-scales over which other variables fluctuate.

Time scales $$\boxed{t_{micro} \ll t \ll t_q}$$

t_q could be the diffusion time-scale for magnetic impurities, the magnetic moments of which will fluctuate in a **magnetic system** or ;

the flipping time of impurities that create random fields acting on other magnetic variables.

Weak disorder (modifies the critical properties but not the phases) *vs.* **strong disorder** (modifies both).

E.g., **random ferromagnets** ($J_{ij} > 0$) *vs.* **spin-glasses** ($J_{ij} \gtrless 0$).

Rugged free-energy landscapes

Glassy physics : beyond the $\lambda\phi^4$ Ginzburg-Landau Questions !

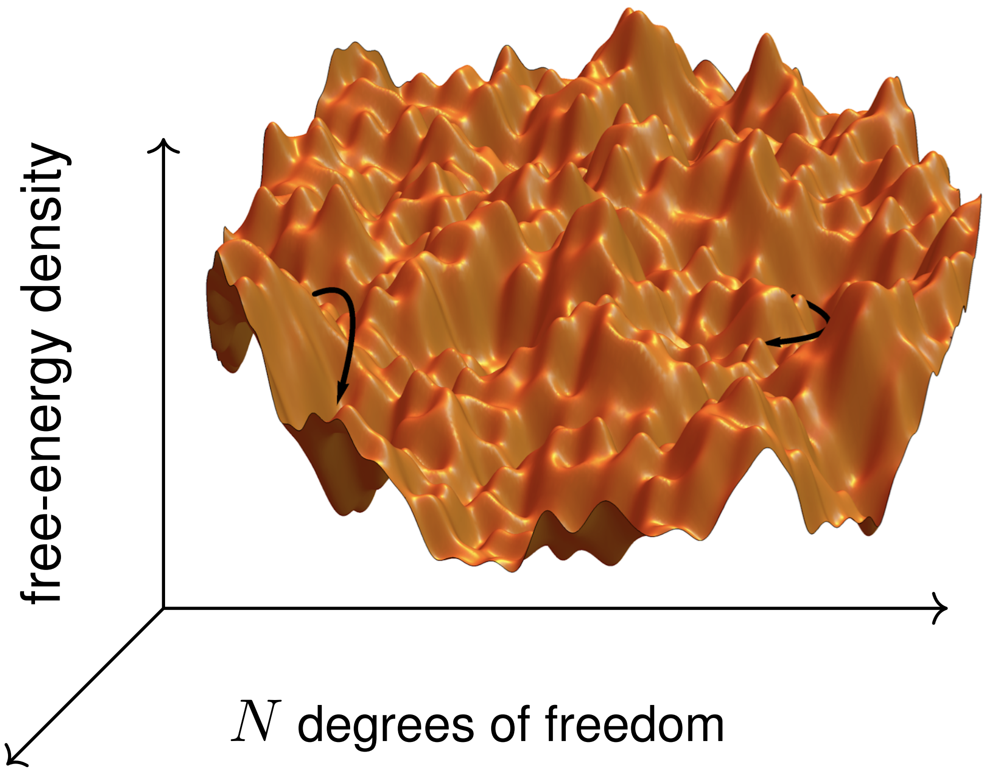


Figure adapted from a picture by **C. Cammarota**

Topography of the landscape on the N-dimensional substrate made by the N order parameters ?

Numerous studies by **theoretical physicists** and **probabilists**

Rugged free-energy landscapes

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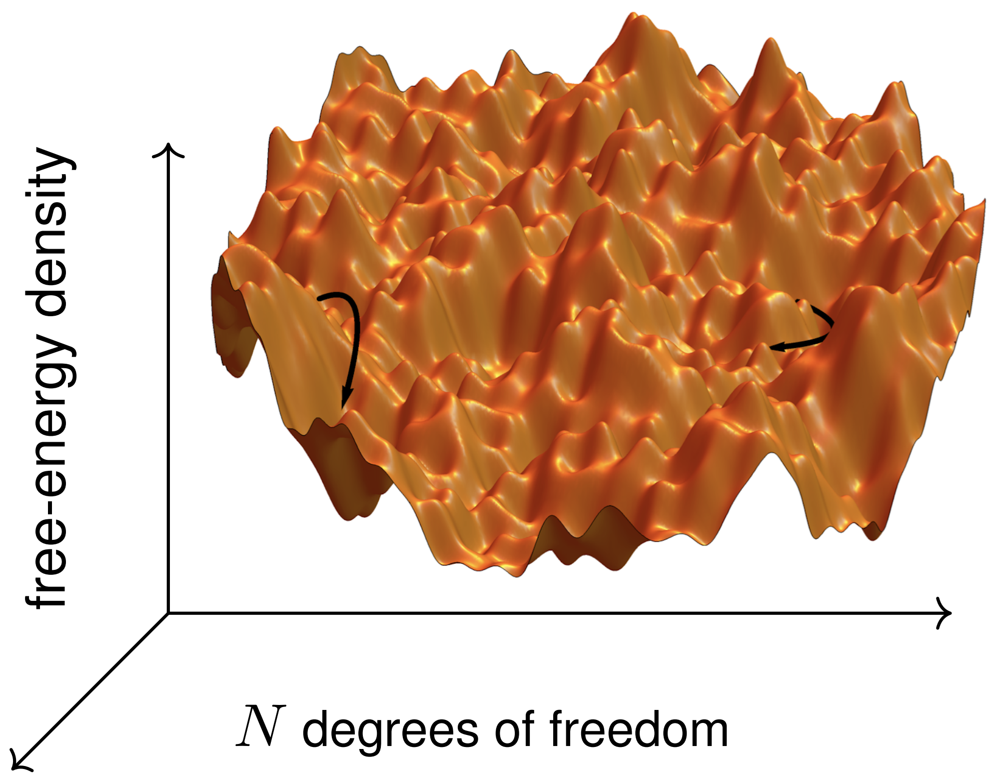


How to reach the absolute minimum?

Thermal activation, surfing over tilted regions, quantum tunneling?

Optimisation problem Smart algorithms? Computer sc - applied math

Spin-glasses

Magnetic impurities (spins) randomly placed in an inert host

Quenched random interactions

Interacting via the RKKY potential

$$V(r) \propto \frac{\sin 2\pi k_F r}{r^3}$$

very rapid oscillations (change in sign) and slow power law decay

Standard lore : there is a 2nd order static phase transition at T_s

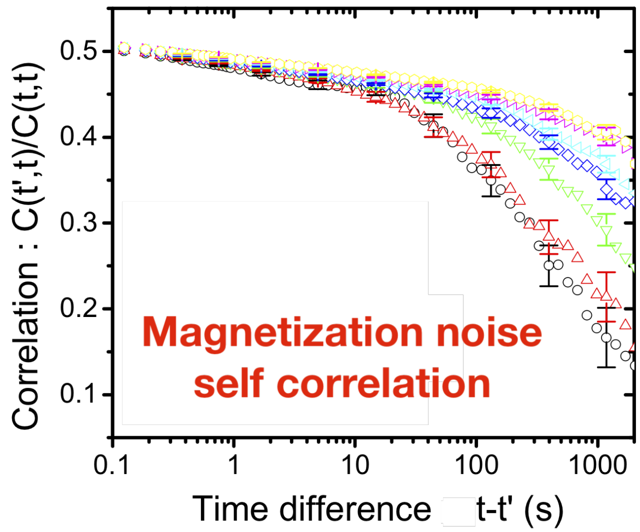
separating a paramagnetic from a spin-glass phase.

No dynamic precursors above T_s.

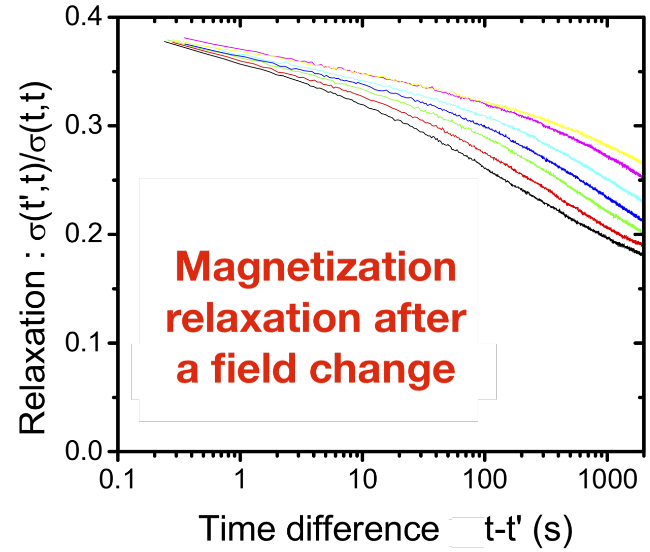
Glassy dynamics below T_s with aging, memory effects, etc.

Rugged free-energy landscapes

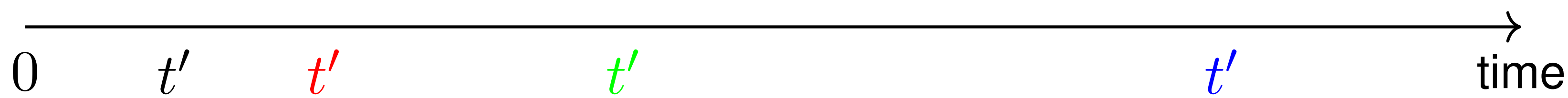
Glassy physics : slow relaxation & loss of stationarity (aging)



Correlation



Linear response

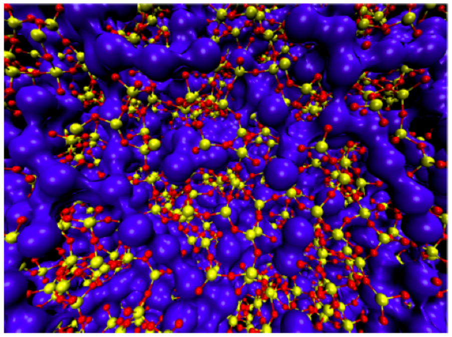


Different curves are measured after log-spaced reference times t' after the quench : **breakdown of stationarity** $\Longrightarrow$ far from equilibrium

No identifiable growing length $\mathcal{R}(t)$: **glassy microscopic mechanisms ?**

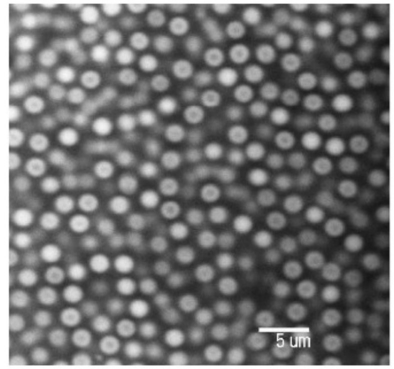
Spin-glass experiments **Hérisson & Ocio 02-04**

What do glasses look like ?



Simulation

Molecular (Sodium Silicate)

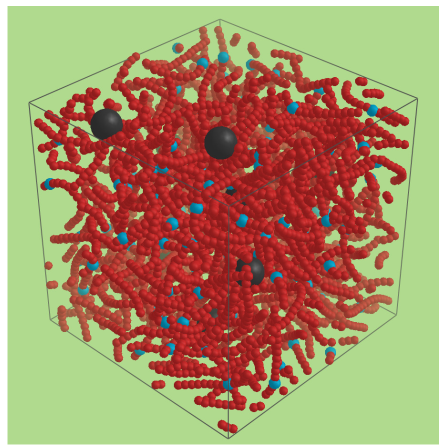


Confocal microscopy

Colloids (e.g. $d \sim 162$ nm in water)

Experiment

Granular matter



Simulation

Polymer melt

Structural Glasses

Characteristics

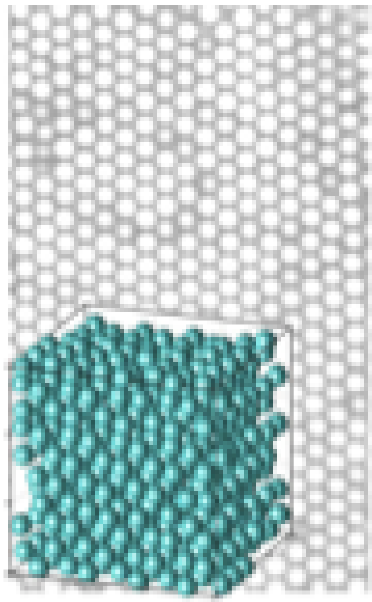
- Selected variables (molecules, colloidal particles, vortices or polymers in the pictures) are coupled to their surroundings (other kinds of molecules, water, etc.) that act as thermal baths in equilibrium.
- There is no quenched disorder.
- The interactions each variable feels are still in competition, e.g. Lenard-Jones potential, frustration.
- Each variable feels a different set of forces, time-dependent heterogeneity.

Sometimes one talks about **self-generated disorder**.

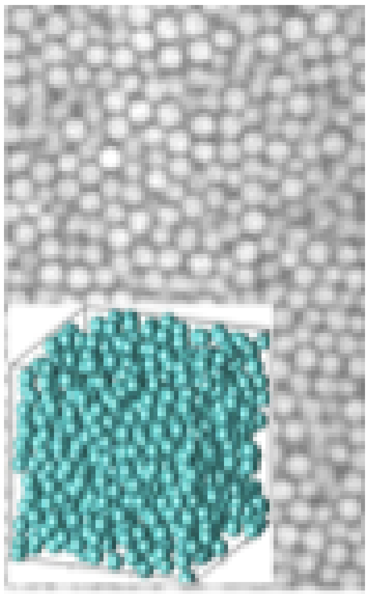
Structural Glasses

e.g., **colloidal ensembles**

Micrometric spheres immersed in a fluid



Crystal



Glass

In the glass : no obvious growth of order, slow dynamics with, however, scaling properties.

What drives the slowing down ?

Correlation functions

One can define a local density $\rho(\boldsymbol{x},t) = N^{-1}\sum_i \delta(\boldsymbol{x} - \boldsymbol{r}_i(t))$ self-correlation

$$\langle\, \rho(\boldsymbol{x},t)\rho(\boldsymbol{y},t_w)\,\rangle$$

The angular brackets indicate a "noise" average ; i.e.

over different dynamical histories (runs of simulation/experiment)

Upon averaging one expects :

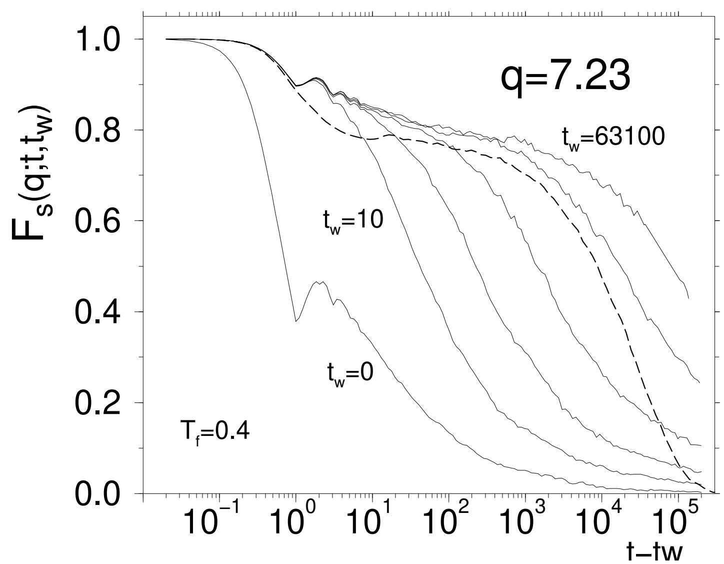
isotropy (all directions are equivalent)

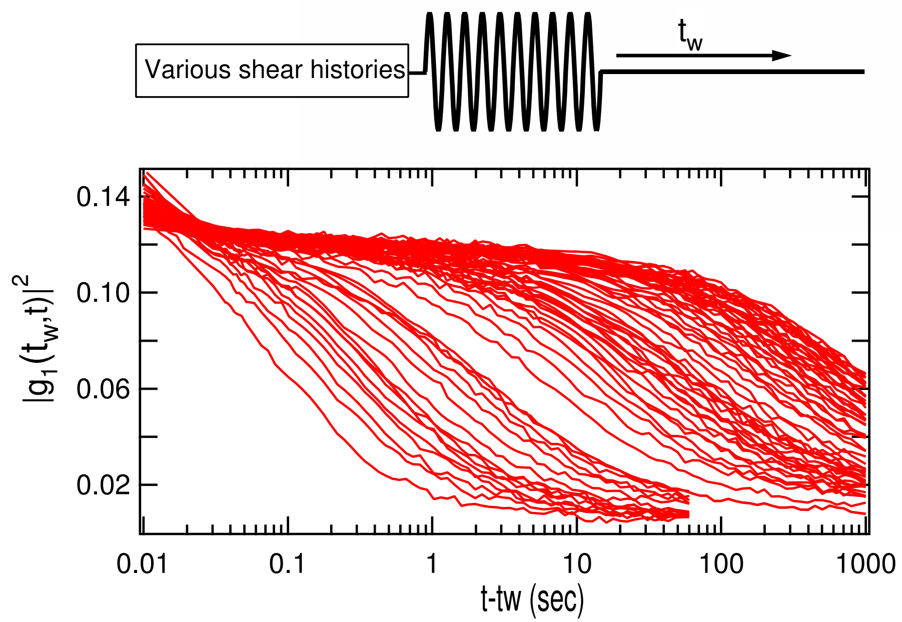
invariance under translations of the reference point

Thus, $\langle\, \rho(\boldsymbol{x},t)\rho(\boldsymbol{y},t_w)\,\rangle \Rightarrow g(r;t,t_w)$, with $r = |\boldsymbol{x} - \boldsymbol{y}|$. Its Fourier transform is $F(q;t,t_w)$ and it has a self part $F_s(q;t,t_w)$ that at equal times becomes the structure factor

Low temp/high densities

Out of equilibrium relaxation





L-J mixture **J-L Barrat & Kob 99**

Colloids **Viasnoff & Lequeux 03**

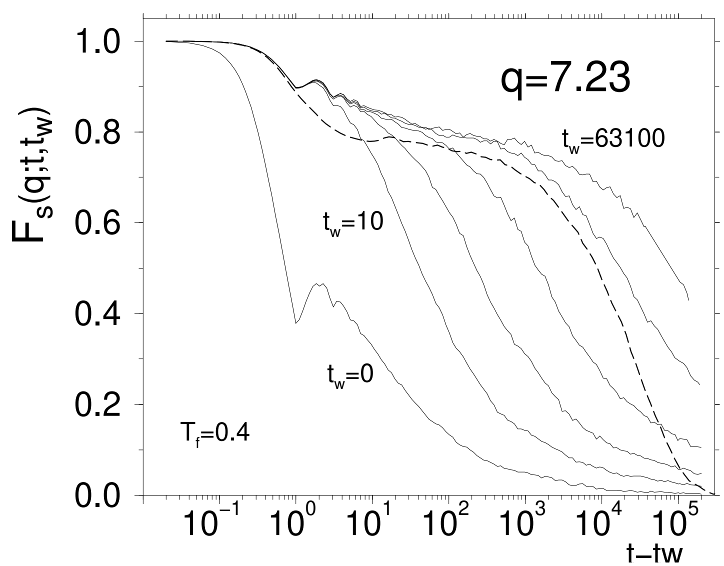
$$t_{micro} \ll t \ll t_{\rm eq}$$

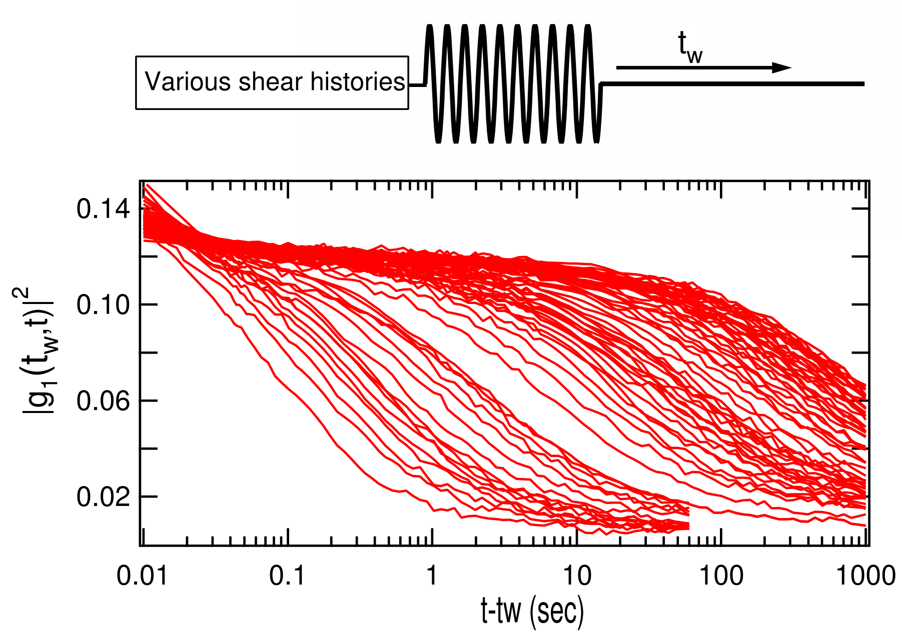
The equilibration time goes beyond the experimentally accessible times

Similar curves found in all other glasses.

Low temp/high densities

Ageing effects





L-J mixture **J-L Barrat & Kob 99**

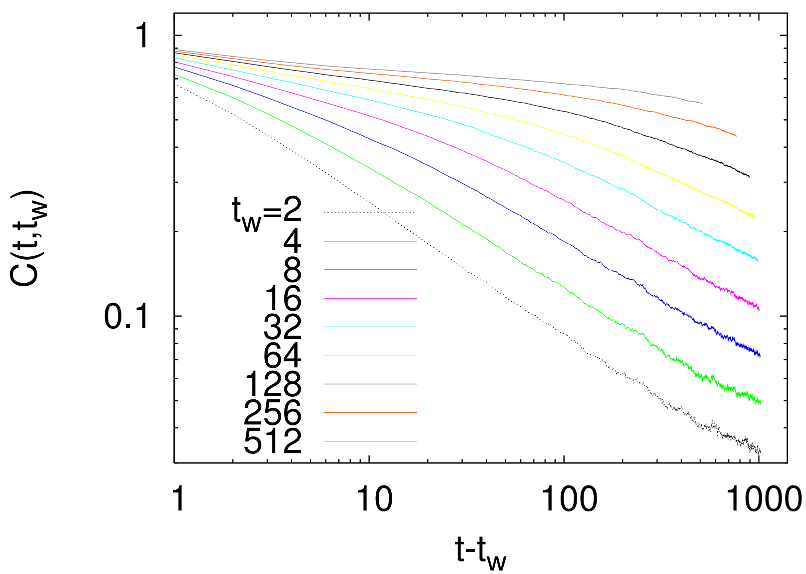
Colloids **Viasnoff & Lequeux 03**

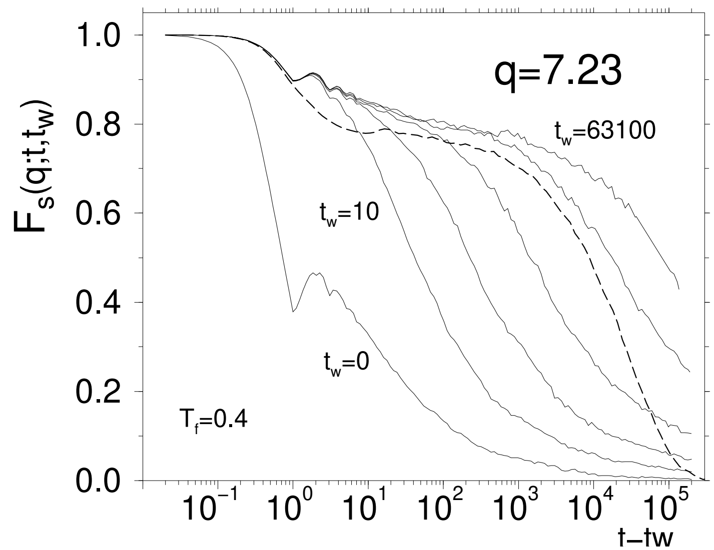
$$t_{micro} \ll t \ll t_{\rm eq}$$

Ageing the relaxation is slower for older systems

Ferromagnet vs glass

Not so different as long as correlations are concerned





2d Ising model - spin-spin

Sicilia *et al.* **07**

Lennard-Jones - density-density

Kob & Barrat 99

One correlation can exhibit stationary and non stationary relaxation

in different two-time regimes

Long time-scales

for relaxation

Systems with competing interactions remain out of equilibrium and it is not clear

- whether there are phase transitions,
- which is the nature of the putative ordered phases,
- which is the dynamic mechanism.

Examples are :

- systems with quenched disorder,
- systems with geometric frustration,
- glasses of all kinds.

Static and dynamic mean-field theory has been developed – both classically and quantum mechanically – and they yield new concepts and predictions.

Extensions of the RG have been proposed and are currently being explored.

Out of equilibrium

Three possible reasons

- The equilibration time goes beyond the experimentally accessible times in **macroscopic systems** in which $t_{\rm eq}$ grows with the system size,

$$\lim_{N\gg 1} t_{\rm eq}(N) \gg t$$

e.g., **diffusion, critical slowing down, coarsening, glassy physics**

- **Driven systems** Energy injection

$$\boldsymbol{F}_{\rm ext} \neq -\boldsymbol{\nabla} V(\boldsymbol{x})$$

e.g., **active matter**

- **Integrability**

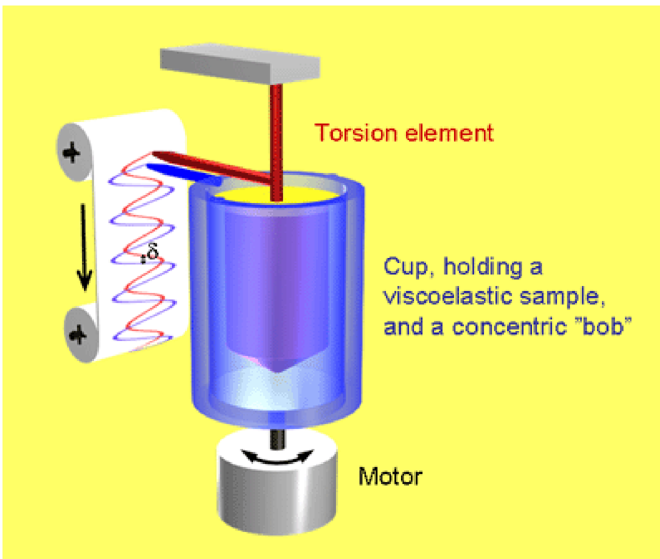
$$I_\mu(\{\boldsymbol{p}_i, \boldsymbol{x}_i\}) = ct, \qquad \mu = 1, \dots, N$$

Too many constants of motion inhibit equilibration to the Gibbs ensembles.

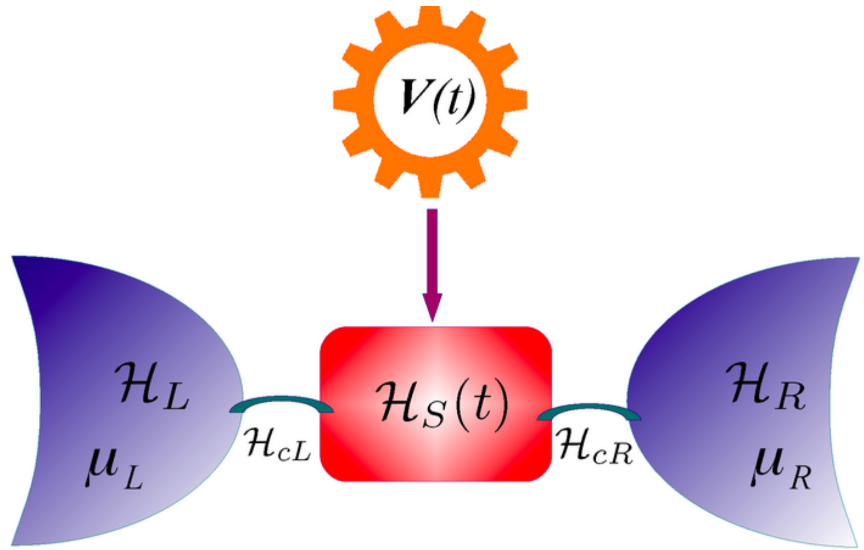
e.g., **1d bosonic gases**

Energy injection

Traditional : from the borders (outside)



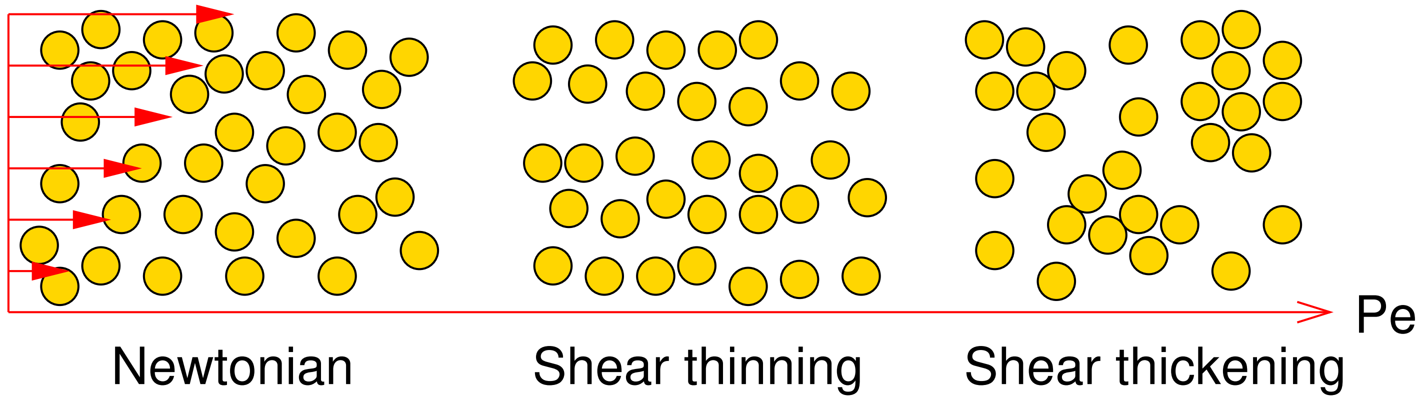
Rheology



Transport

Drive & transport

Rheology of complex fluids



Rheology of complex fluids

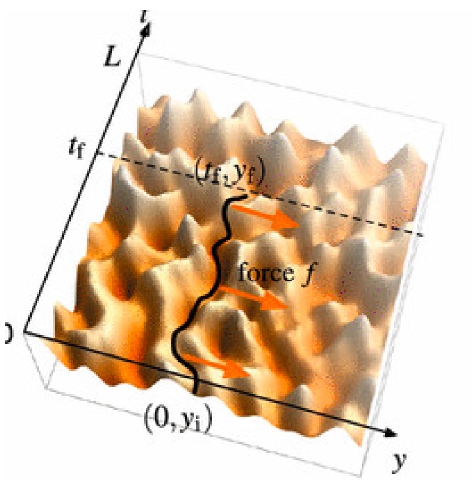
Shear thinning τ_{relax} decreases, *e.g.* paints

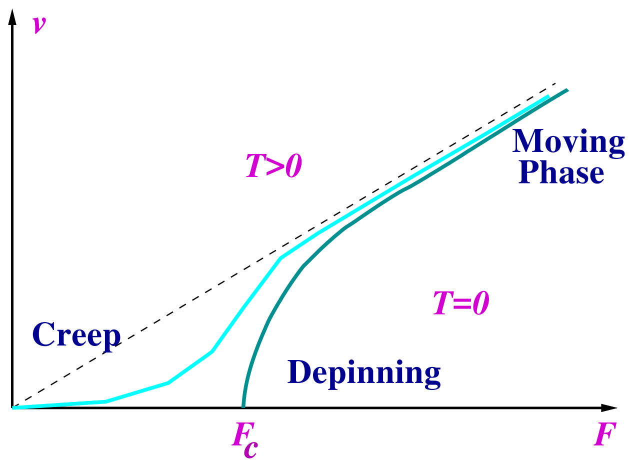
Shear thickening τ_{relax} increases, *e.g.* cornstarch & water mix

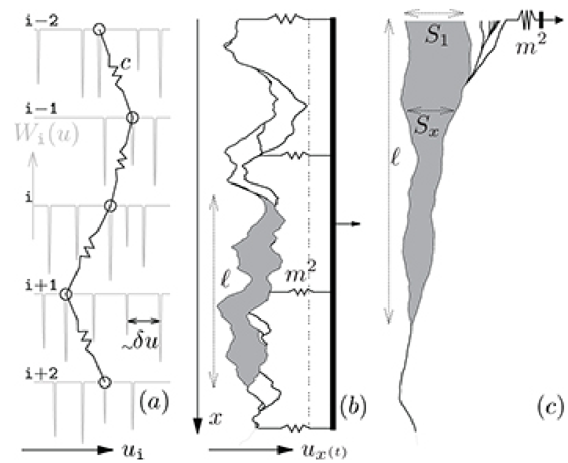
e.g. review **Brader 10**

Drive & transport

Driven interface over a disordered background







A line — Depinning & creep — avalanches

e.g. review **Giamarchi et al 05**, connections to earthquakes **Landes 16**

Active matter

Definition

Active matter is composed of large numbers of active "agents", each of which consumes energy in order to move or to exert mechanical forces.

Due to the energy consumption, these systems are intrinsically out of thermal equilibrium.

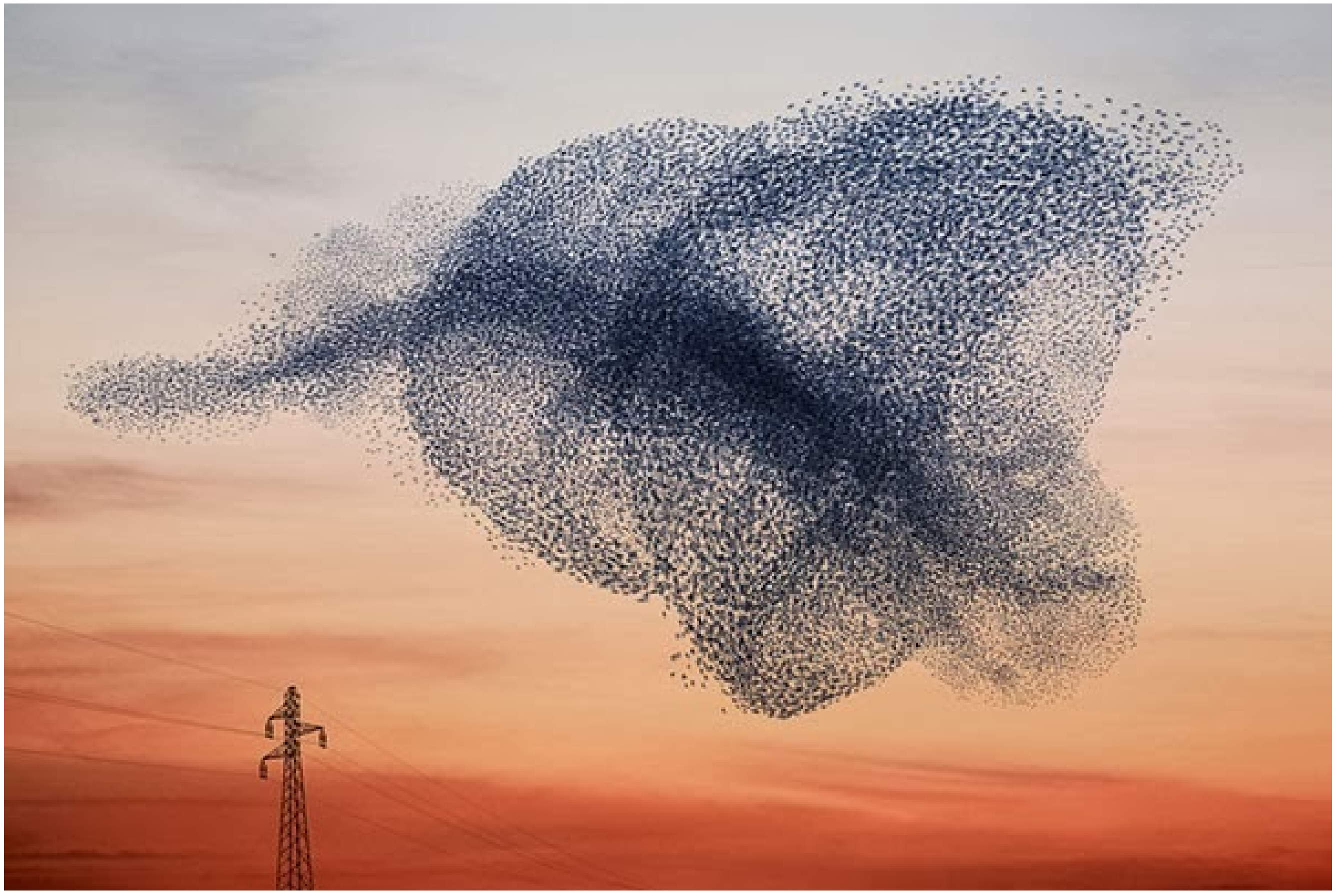
Energy injection is done “uniformly” within the samples (and not from the borders).

Coupling to the environment (bath) allows for the dissipation of the injected energy.

Fourth Lecture

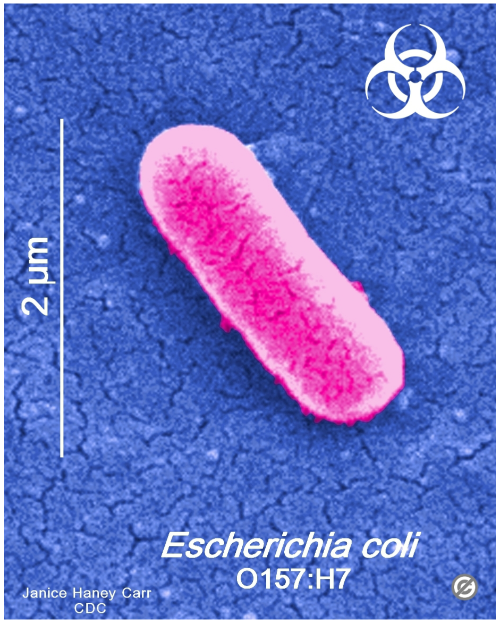
Natural systems

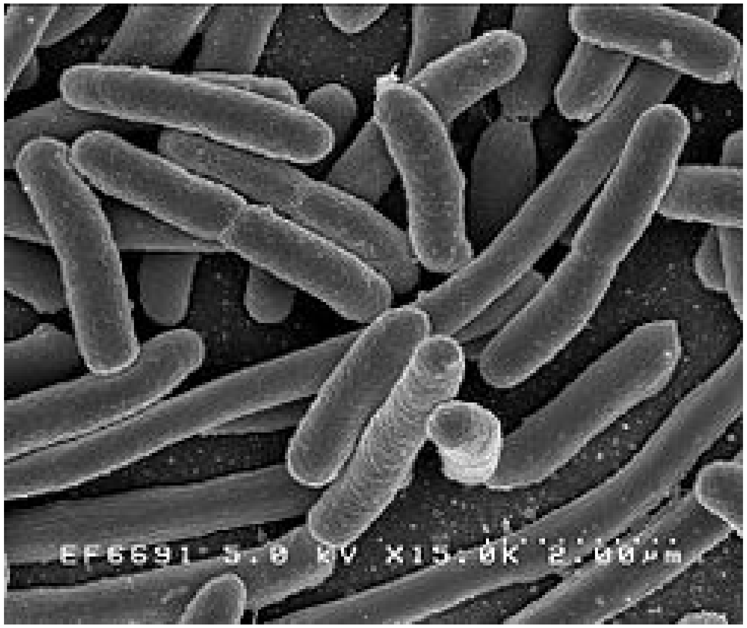
Birds flocking



Natural systems

Bacteria

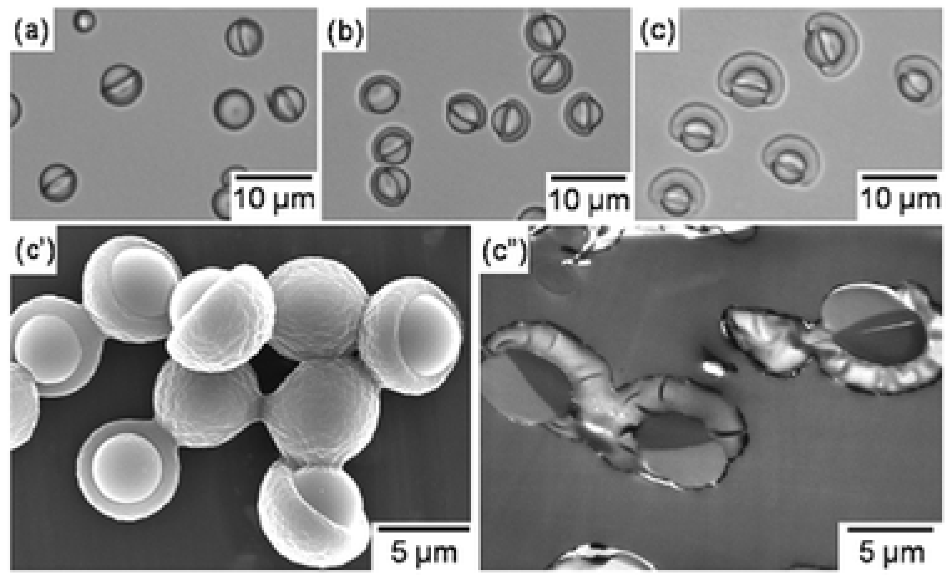




Escherichia coli - Pictures borrowed from the internet.

Artificial systems

Janus particles



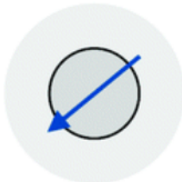
Particles with two faces (Janus God)

e.g. **Bocquet group** ENS Lyon-Paris, **di Leonardo group** Roma

Active Brownian particles

The standard model – ABPs

Spherical particles with diameter σ_d



Environment $\Longrightarrow$ Langevin dynamics

Scales $\Longrightarrow$ over-damped motion

Self-propulsion $\Longrightarrow$ active force $\boldsymbol{F}_{\rm act}$ along $\boldsymbol{n}_i = (\cos\theta_i(t), \sin\theta_i(t))$

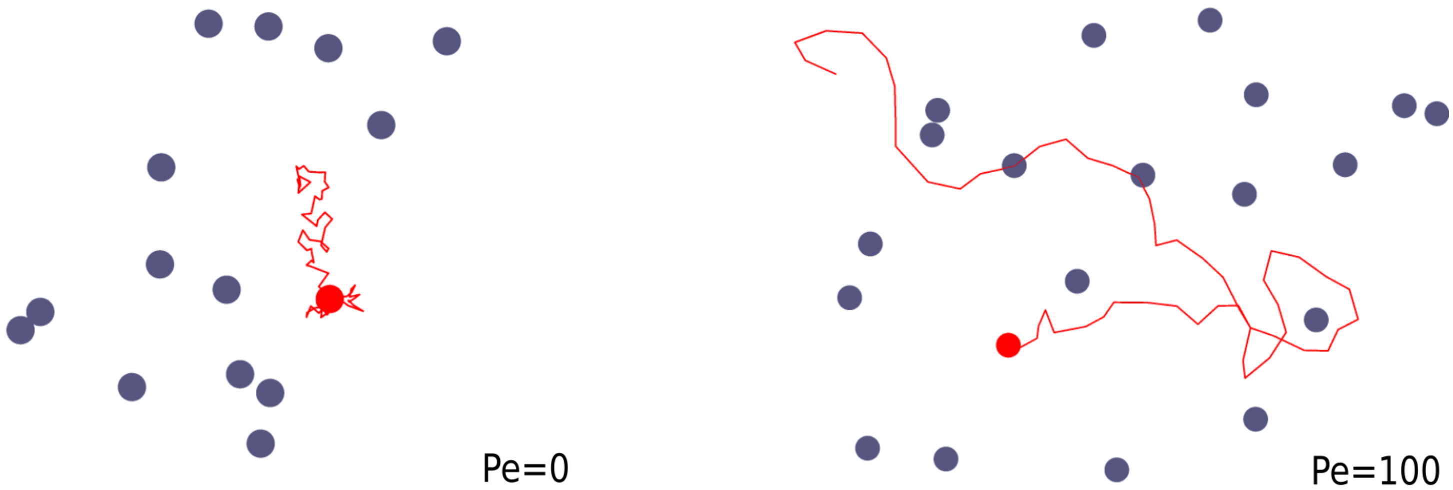
$$\underbrace{\gamma\dot{\boldsymbol{r}}_i}_{\text{friction}} = \underbrace{F_{\rm act}\boldsymbol{n}_i}_{\text{propulsion}} \underbrace{- \vec{\nabla}_i \sum_{j(\neq i)} U(r_{ij})}_{\text{inter-particle repulsion}} + \underbrace{\boldsymbol{\xi}_i}_{\text{translational white noise}} \qquad \underbrace{\dot{\theta}_i = \eta_i}_{\text{rotational white noise}}$$

$2d$ packing fraction $\phi = \pi\sigma_d^2 N/(4S)$ Péclet number Pe = $F_{\rm act}\sigma_{\rm d}/(k_B T)$

Bialké, Speck & Löwen 12, Fily & Marchetti 12

Active Brownian particles

Typical motion of ABPs in interaction



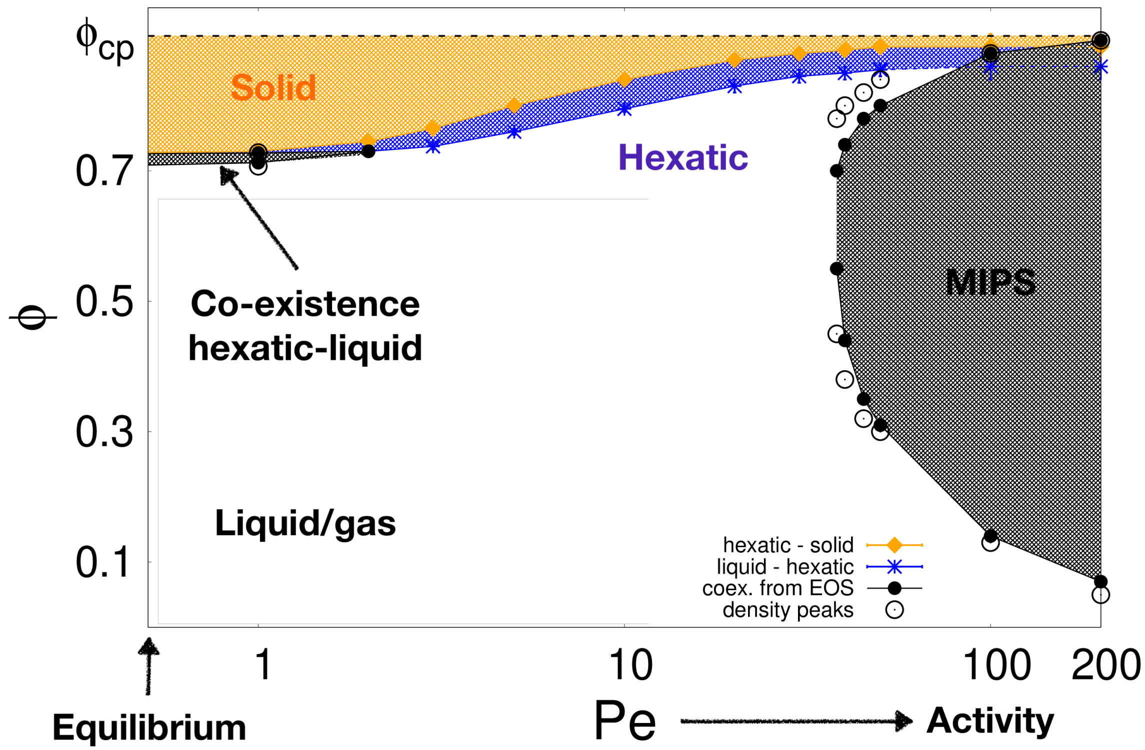
The activity induces a persistent random motion

Long running periods $\ell_p \propto \text{Pe}\ \sigma_d$ and

sudden changes in direction

Active Brownian particles

Complex out of equilibrium phase diagram



Motility induced phase separation (MIPS)

gas & dense droplet

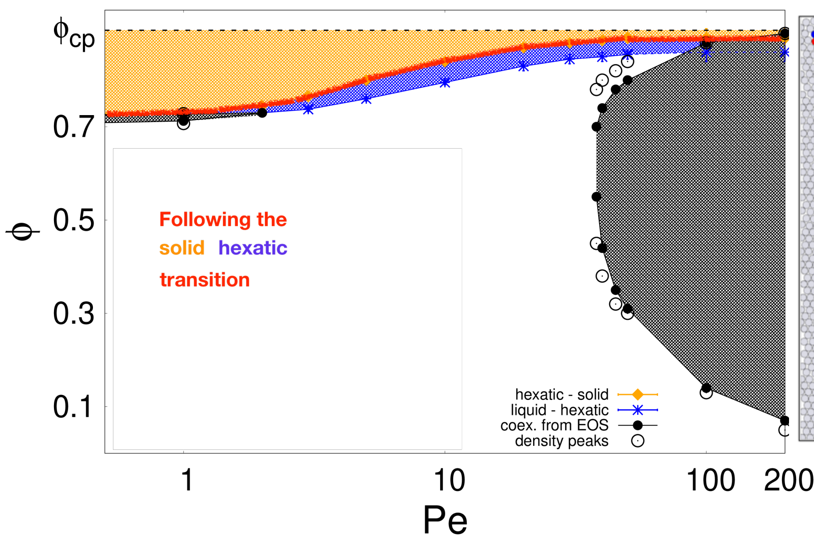
Cates & Tailleur 12

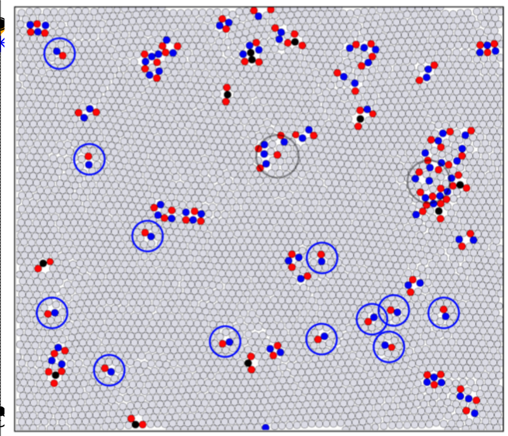
From virial pressure $P(\phi)$, translational and orientational correlations G_T and G_6, distributions of local density and hexatic order ϕ_i and ψ_{6i}, at fixed $k_BT = 0.05$

Digregorio, Levis, Suma, LFC, Gonnella & Pagonabarraga 18

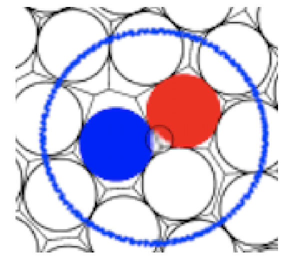
Active Brownian particles

Out of equilibrium phase diagram **First question (out of many !)**





Free dislocation:
a 7-5 neighbor



$\neq$ from $\triangle$ lattice

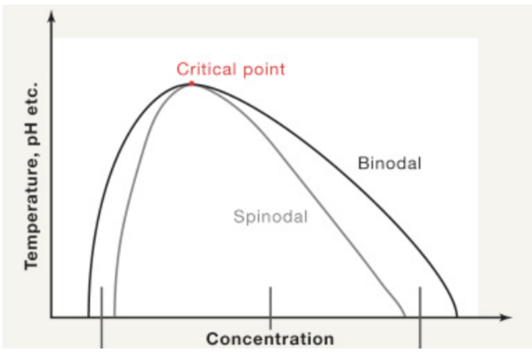
Solid - **Hexatic** transition at ϕ_{sh}, **driven by unbinding of dislocation pairs as in Berezinskii-Kosterlitz-Thouless-Halperin-Nelson-Young universality ?**

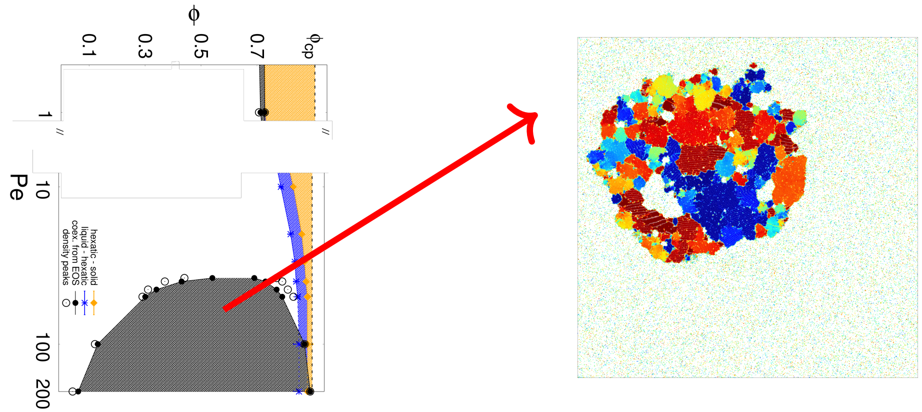
$$\rho_{disloc} \simeq a \, \exp\left[-b\left(\frac{\phi_{sh}}{\phi_{sh}-\phi}\right)^{\nu}\right] \qquad \nu \sim 0.37 \quad \forall \text{Pe} \, ?$$

Digregorio, Levis, LFC, Gonnella & Pagonabarraga 21

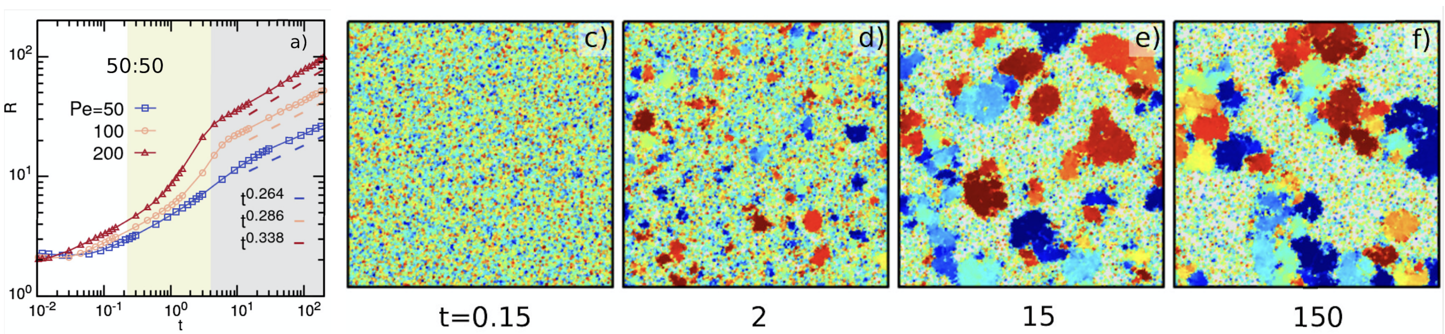
Active Brownian particles

Out of equilibrium phase diagram So many questions !





Dynamics of formation of the dense phase ? but bubbles, hexatic order, ...



Universality with the Lifshitz-Slyozov law $\mathcal{R}(t) \simeq t^{1/3}$? Geometry ?

Redner *et al* **13, Stenhammar** *et al* **14, ... , Caporusso** *et al* **20, Caprini** *et al* **20, ...**

Out of equilibrium

Three possible reasons

- The equilibration time goes beyond the experimentally accessible times in **macroscopic systems** in which $t_{\rm eq}$ grows with the system size,

$$\lim_{N\gg 1} t_{\rm eq}(N) \gg t$$

e.g., **diffusion, critical slowing down, coarsening, glassy physics**

- **Driven systems** Energy injection

$$\boldsymbol{F}_{\rm ext} \neq -\boldsymbol{\nabla} V(\boldsymbol{x})$$

e.g., **active matter**

- **Integrability**

$$I_\mu(\{\boldsymbol{p}_i, \boldsymbol{x}_i\}) = ct, \qquad \mu = 1, \dots, N$$

Too many constants of motion inhibit equilibration to the Gibbs ensembles

e.g., **1d bosonic gases**

Questions

Does an isolated quantum system reach some kind of equilibrium ?

Boosted by recent interest in

— the dynamics after quantum quenches of cold atomic systems

rôle of interactions (integrable *vs.* non-integrable)

— many-body localisation

novel effects of quenched disorder

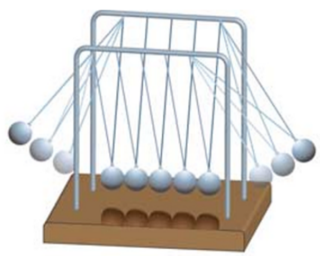
And, an isolated classical system ?

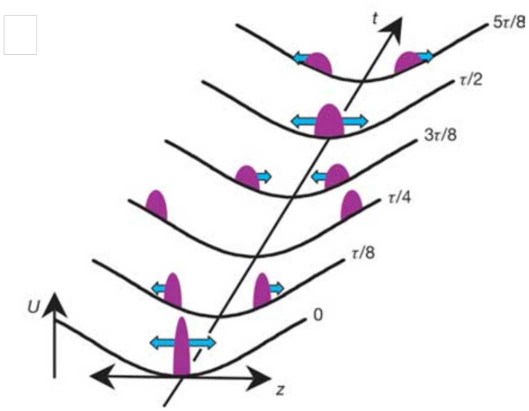
The (old) ergodicity question revisited

Our contribution **Barbier, LFC, Lozano, Nessi, Picco, Tartaglia 17-21**

Motivation

Isolated quantum systems: experiments and theory $\sim$ 15y ago





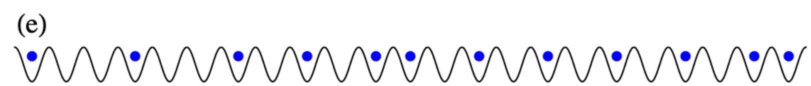
A quantum Newton's cradle
cold atoms in isolation

Kinoshita, Wenger & Weiss 06

Quantum quenches & Conformal field theory

Calabrese & Cardy 06

Numerics of lattice hard core bosons



Rigol, Dunjko, Yurovsky & Olshanii 07

and many others

$1d$ lattice models & 1+1 field theories

Alba, Bernard, Bertini, Calabrese, Cardy, Caux, De Luca, De Nardis, Doyon, Essler, Dubail, Gambassi, Konik, Mussardo, Polkovnikov, Prosen, Silva, Santoro, Spohn...

Quantum quenches

Definition & questions

- Take an isolated quantum system with Hamiltonian $\hat{H}_0$
- Initialize it in, say, $|\psi_0\rangle$ the ground-state of $\hat{H}_0$ (or any $\hat{\rho}(t_0)$)
- Unitary time-evolution $\hat{U} = e^{-\frac{i}{\hbar}\hat{H}t}$ with a Hamiltonian $\hat{H} \neq \hat{H}_0$.

Does the system reach (locally) a steady state?

Are the expected values of local observables determined by $e^{-\beta\hat{H}}$?

Does the evolution occur as in equilibrium ?

Not for integrable models. Alternative, the **Generalized Gibbs Ensemble**

$$\hat{\rho}_{\rm GGE} = \mathcal{Z}^{-1}(\{\gamma_\mu\})\; e^{-\sum_{\mu=1}^{N}\gamma_\mu \hat{I}_\mu} \quad \& \quad \langle\psi_0|\hat{I}_\mu|\psi_0\rangle = \langle\hat{I}_\mu\rangle_{\rm GGE} \text{ fix } \{\gamma_\mu\}$$

Classical quenches

Definition & questions

- Take an **isolated** classical system with Hamiltonian H_0, evolve with H
- Initialize it in, say, ψ_0 a configuration, *e.g.* $\{\boldsymbol{x}_i, \boldsymbol{p}_i\}_0$ for a particle system

  ψ_0 could be drawn from a probability distribution, *e.g.* $\mathcal{Z}^{-1} \, e^{-\beta_0 H_0(\psi_0)}$

Does the system reach a steady state? (in the $N \to \infty$ limit)

Is it described by a thermal equilibrium probability $e^{-\beta H}$?

Do at least some local observables behave as thermal ones?

Does the evolution occur as in equilibrium?

If not, other kinds of probability distributions?

Classical quenches

Definition & questions

In the steady state of a classical macroscopic ($N \to \infty$) model

Time averages $\overline{O(t)} \equiv \lim_{\tau\to\infty} \frac{1}{\tau}\int_{t_{\rm st}}^{t_{\rm st}+\tau} dt'\; O(t')$

& statistical averages $\langle O\rangle \equiv \int \prod_i dx_i \prod dp_i\; O(x_i,p_i)\;\rho(x_i,p_i)$

should be equal $\overline{O(t)} = \langle O\rangle$ for a generalised micro-canonical measure ρ

in which, in integrable cases, all constants of motion are fixed **Yuzbashyan 18**

Are local observables characterised by a "canonical" measure ?

If yes, which ones ?

Classical quenches

Interest in integrable models : strategy & goals

– Choose a sufficiently simple classical *integrable interacting* model ($2N$ phase-space variables, N constants of motion) with an interesting *phase diagram* to investigate different *initial conditions* and *quenches* across the *phase transition(s)*

– Solve the *dynamics* after the quenches

– Build a *Generalised Gibbs Ensemble* (GGE)

– Prove that the asymptotic limit of *local observables* is captured by the GGE

Plan of the 1st Lecture

Plan

1. Equilibrium *vs.* out of equilibrium classical systems.

2. How can a classical system stay far from equilibrium ?

   From single-particle to many-body

   Diffusion

   Phase-separation & domain growth

   Quenched randomness & glasses

   Driven systems

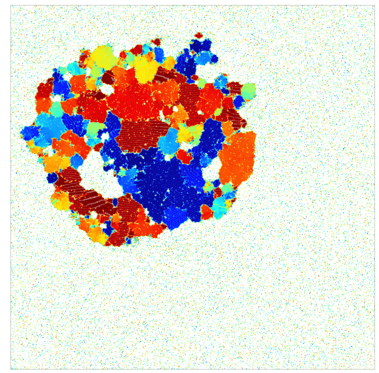
   Active matter

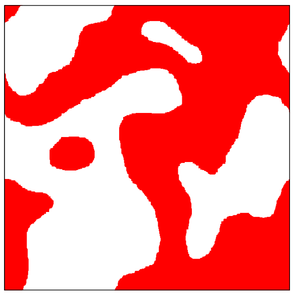
3. **Purposes**

Out of equilibrium

Explain, describe and, something in common ?

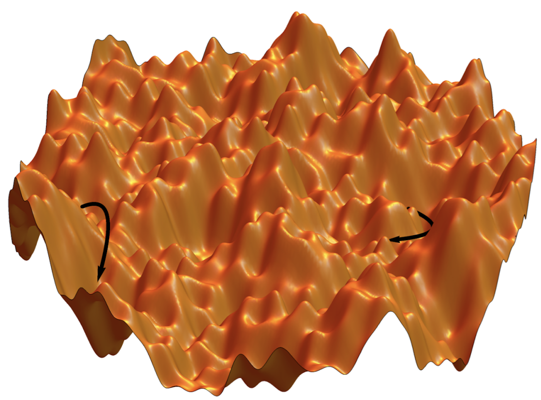
$$\boldsymbol{F} \neq -\boldsymbol{\nabla} V(\boldsymbol{r})$$

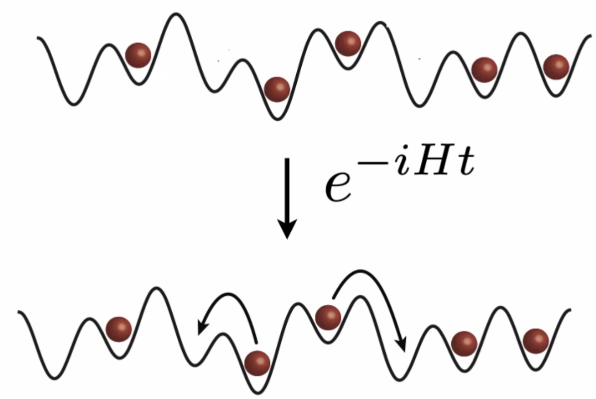




$$\lim_{N \gg 1} t_{\rm eq}(N) \gg t$$

$$I_\mu = \text{ct} \qquad \mu = 1, \dots, N$$





Each lecture will treat one of these systems

Challenges

in classical non-equilibrium macroscopic systems

- Coarsening

  The systems are taken across *usual phase transitions*.

  The *dynamic mechanisms* are well-understood :

  competition between equilibrium phases & topological defect annihilation.

  The difficulty lies in the calculation of observables in a time-dependent non-linear field theory.

- Glasses & active matter

  Are there *phase transitions* ?

  The *dynamic mechanisms* are not well understood.

  The difficulty is conceptual (also computational).

- General question

  Do these enjoy some kind of thermodynamic properties ?

End of 1st Lecture

Methods

Many body systems

- **Coarsening phenomena**

  Identify the order parameter $\phi(\boldsymbol{x}, t)$ (a field). Write Langevin or Fokker-Planck equations for it and analyse them. A difficult problem. Non-linear equations. Neither perturbation theory nor RG methods are OK. Self-consistent resummations tried.

- **Glassy systems**

  The "order parameter" is a composite object depending on two-times. Spin models with quenched randomness yield a mean-field description of the dynamics observed. Classes of systems (ferromagnets, spin-glass and fragile glasses) captured.

- **Active matter**

  Numerics of agent-based models, field theories, expansions...

Observables

Positional order

The (fluctuating) local particle number density

$$\rho(\boldsymbol{r}_0) = \sum_{i=1}^{N} \delta(\boldsymbol{r}_0 - \boldsymbol{r}_i)$$

with normalisation $\int d^d\boldsymbol{r}_0 \, \rho(\boldsymbol{r}_0) = N$.

The density-density correlation function $C(\boldsymbol{r}+\boldsymbol{r}_0, \boldsymbol{r}_0) = \langle \rho(\boldsymbol{r}+\boldsymbol{r}_0)\rho(\boldsymbol{r}_0)\rangle$ that, for homogeneous (independence of $\boldsymbol{r}_0$) and isotropic ($\boldsymbol{r} \mapsto |\boldsymbol{r}| = r$) cases, is simply $C(\boldsymbol{r}+\boldsymbol{r}_0, \boldsymbol{r}_0) = C(r)$.

The double sum in $C(\boldsymbol{r}+\boldsymbol{r}_0, \boldsymbol{r}_0) = \langle \sum_{ij} \delta(\boldsymbol{r}+\boldsymbol{r}_0 - \boldsymbol{r}_i)\delta(\boldsymbol{r}_0 - \boldsymbol{r}_j)\rangle$ has contributions from $i = j$ and $i \neq j$: $C_{\rm self} + C_{\rm diff}$

Observables

Positional order

The density-density correlation function

$$C(\boldsymbol{r}+\boldsymbol{r}_0, \boldsymbol{r}_0) = \langle \rho(\boldsymbol{r}+\boldsymbol{r}_0)\rho(\boldsymbol{r}_0)\rangle = \sum_{ij} \langle \delta(\boldsymbol{r}+\boldsymbol{r}_0-\boldsymbol{r}_i)\delta(\boldsymbol{r}_0-\boldsymbol{r}_i)\rangle$$

is linked to the structure factor

$$S(\boldsymbol{q}) \equiv N^{-1}\langle \tilde{\rho}(\boldsymbol{q})\tilde{\rho}(-\boldsymbol{q})\rangle = \frac{1}{N}\langle \sum_{i=1}^{N}\sum_{j=1}^{N} e^{-\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}\rangle$$

by

$$N\, S(\boldsymbol{q}) \;=\; \int d^d r_1 \int d^d r_2\, C(\boldsymbol{r}_1, \boldsymbol{r}_2)\, e^{-\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)}$$

Observables

Positional order

In isotropic cases, i.e. liquid phases, the pair correlation function

$\frac{N}{V}\, g(r) =$ average number of particles at distance r

from a tagged particle at $\boldsymbol{r}_0$

is linked to the structure factor

$$S(\boldsymbol{q}) = \frac{1}{N} \langle \sum_{i=1}^{N} \sum_{j=1}^{N} e^{-\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_j)} \rangle$$

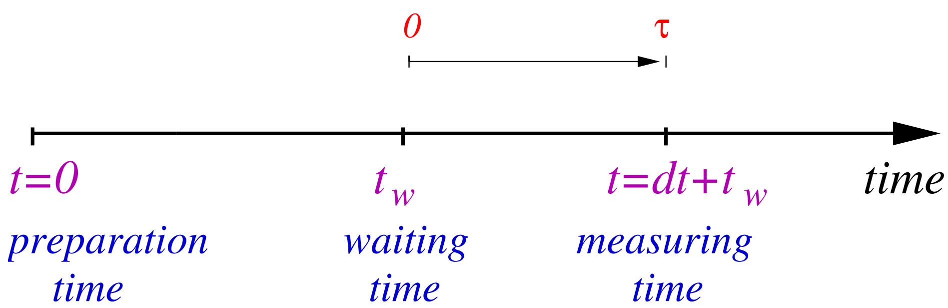
by

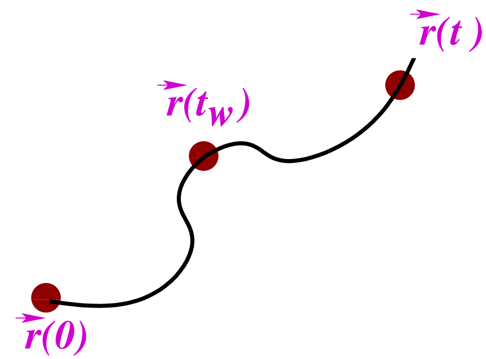
$$S(\boldsymbol{q}) = 1 + \frac{N}{V} \int d^d r\; g(r) e^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}$$

Peaks in $g(r)$ are related to peaks in $S(q)$. The first peak in $S(q)$ is at $q_0 = 2\pi/\Delta r$ where Δr is the distance between peaks in $g(r)$ (that is close to the inter particle distance as well).

Two-time observables

Correlations





t_w not necessarily longer than t_{eq}.

The two-time correlation between $A[\{\boldsymbol{r}_i(t)\}]$ and $B[\{\boldsymbol{r}_i(t_w)\}]$ is

$$C_{AB}(t, t_w) \equiv \langle\, A[\{\boldsymbol{r}_i(t)\}] B[\{\boldsymbol{r}_i(t_w)\}]\,\rangle$$

average over realizations of the dynamics (initial conditions, random numbers in a MC simulation, thermal noise in Langevin dynamics, etc.)

Correlation functions

One can define a two-time dependent density-density correlation

$$\langle \rho(\boldsymbol{x}, t)\rho(\boldsymbol{y}, t_w) \rangle$$

The angular brackets indicate a “thermal” average ; i.e.

over different dynamical histories (runs of simulation/experiment)

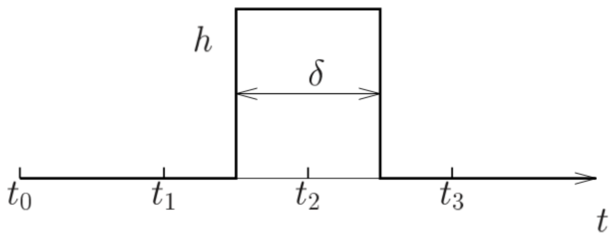
Upon averaging one expects :

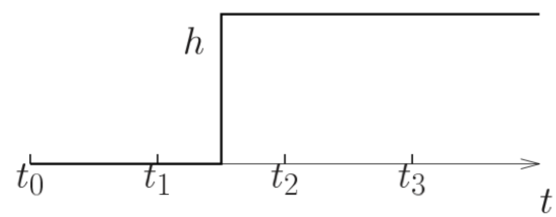
isotropy (all directions are equivalent)

invariance under translations of the reference point $\boldsymbol{x}$.

Thus, $\langle \rho(\boldsymbol{x}, t)\rho(\boldsymbol{y}, t_w) \rangle \Rightarrow g(r; t, t_w)$, with $r = |\boldsymbol{x} - \boldsymbol{y}|$. Its Fourier transform is $F(q; t, t_w)$ and it has a self part $F_s(q; t, t_w)$ that at equal times becoes the structure factor

Response to perturbations





The perturbation couples linearly to the observable $B[\{\boldsymbol{r}_i\}]$

$$H \quad \to \quad H - hB[\{\boldsymbol{r}_i\}]$$

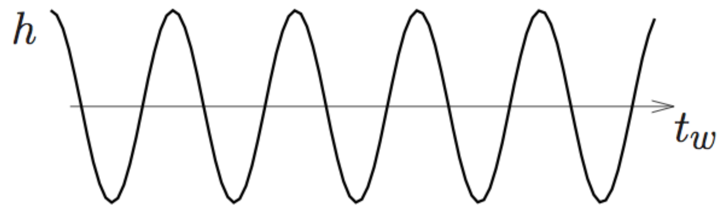
The linear instantaneous response of another observable $A(\{\boldsymbol{r}_i\})$ is

$$R_{AB}(t, t_w) \equiv \left\langle \left. \frac{\delta A[\{\boldsymbol{r}_i\}](t)}{\delta h(t_w)} \right|_{h=0} \right\rangle$$

The linear integrated response or dc susceptibility is

$$\chi_{AB}(t, t_w) \equiv \int_{t_w}^{t} dt' \, R_{AB}(t, t')$$

ac response to perturbations



$$\chi(\omega, t_w) = \int_0^{t_w} dt'\, R(t_w, t')h(\omega, t') = \int_0^{t_w} dt'\, R(t_w, t')e^{i\omega t'}$$

$\chi'(\omega, t_w) = \text{Re}\chi(\omega, t_w)$ (in phase)

$\chi''(\omega, t_w) = \text{Im}\chi(\omega, t_w)$ (out of phase)

are related by Kramers-Krönig $\chi''(\omega, t_w) = -\pi^{-1} P \int d\omega' \, \frac{\chi'(\omega, t_w)}{\omega' - \omega}$

In equilibrium $\chi(\omega, t_w) \to \chi(\omega)$

Disordered spin systems

Classical p-spin model

$$H_{syst} = - \sum_{i_1 < \dots < i_p}^{N} J_{i_1 i_2 \dots i_p} s_{i_1} s_{i_2} \dots s_{i_p}$$

Ising, $s_i = \pm 1$, or spherical, $\sum_{i=1}^N s_i^2 = N$, spins.

Sum over all p-uplets on a complete graph : fully-connected model.

Random exchanges $P(J_{i_1 i_2 \dots i_p}) = e^{-\frac{1}{2} J^2_{i_1 i_2 \dots i_p} (2N^{p-1}/(p! J^2))}$

Extensions to random graphs possible : dilute models.

$p = 2$ Ising : Sherrington-Kirkpatrick model for spin-glasses

$p = 2$ spherical $\approx$ mean-field ferromagnet

$p \geq 3$ Ising or spherical : models for fragile glasses

Methods

for classical and quantum disordered systems

Statics

TAP Thouless-Anderson-Palmer Replica theory	fully-connected (complete graph) Gaussian approx. to field-theories
Cavity or Peierls approx.	dilute (random graph)
Bubbles & droplet arguments functional RG	finite dimensions

Dynamics

Generating functional for classical field theories (MSRJD).

Schwinger-Keldysh closed-time path-integral for quantum dissipative models (the previous is recovered in the $\hbar \to 0$ limit).

Perturbation theory, renormalization group techniques, self-consistent approx.

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Some references

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