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# Dynamics of disordered systems

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# Plan of Lectures

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1. Introduction
2. **Coarsening processes**
3. Formalism
4. Dynamics of disordered spin models



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# References

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— **Phase ordering kinetics & critical dynamics**

A. J. Bray, *Theory of phase ordering kinetics*, Adv. Phys. **43**, 357 (1994).

S. Puri, *Kinetics of Phase Transitions*, (Vinod Wadhawan, 2009).

L. F. Cugliandolo, *Topics in coarsening phenomena*, arXiv:0911.0771, Physica A 389, 4360 (2010).

F. Corberi & A. Politi eds., *Coarsening dynamics*, Comptes Rendus Physique **16** (2015).

P. Krapivsky, S. Redner and E. Ben-Naim, *A kinetic view of statistical physics* (Cambridge University Press, 2010).

M. Henkel and M. Pleimling, *Non-Equilibrium Phase Transitions : Volume 2 : Ageing and Dynamical Scaling Far from Equilibrium*, (Springer, 2010).

L. F. Cugliandolo, *Dynamics of glassy systems*, Les Houches Session 77, arXiv :cond-mat/0210312.

U. C. Tauber, *Critical Dynamics : A Field Theory Approach to Equilibrium and Non-Equilibrium Scaling Behavior* (Cambridge University Press, 2014)

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# Plan of the lecture

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1. The phenomenon
2. Theoretical setting
3. Critical and sub-critical quenches
4. Dynamic scaling
5. Dynamic universality classes
6. Two-time correlations and ageing
7. Two-time responses and loss of memory
8. Mean-field models
9. Modern studies

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# Phenomenon

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The talk focuses on a very well-known example

## Dynamics following a change of a control parameter

- If there is an equilibrium phase transition, the **equilibrium phases** are known on both sides of the transition.  
i.e. the asymptotic state is known.
- For a purely dynamic problem, the **absorbing states** are known.
- The **dynamic mechanism** towards equilibrium (or the absorbing states) is understood the systems try to order locally in one of the few competing states.

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# Interests and goals

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**Practical** interest, *e.g.*

- Mesoscopic structure effects on the opto-mechanical properties of phase separating glasses
- Cooling rate effects on the density of topological defects in cosmology and condensed matter

**Fundamental** interest, *e.g.*

- A theoretical problem beyond perturbation theory.
- Are there growth phenomena in problems with yet unknown dynamic mechanisms? **e.g. glasses**
- Generic features of macroscopic systems out of equilibrium?

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# Context

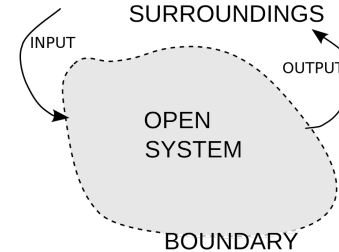
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## Open systems

Our interest is to describe the **dynamics** of a **classical** (or quantum) **system** coupled to a **classical** (or quantum) **environment**.

The Hamiltonian of the ensemble is

$$H = H_{syst} + H_{env} + H_{int}$$



The dynamics of all variables are given by **Newton** (or Heisenberg) rules, depending on the variables being classical (or quantum).

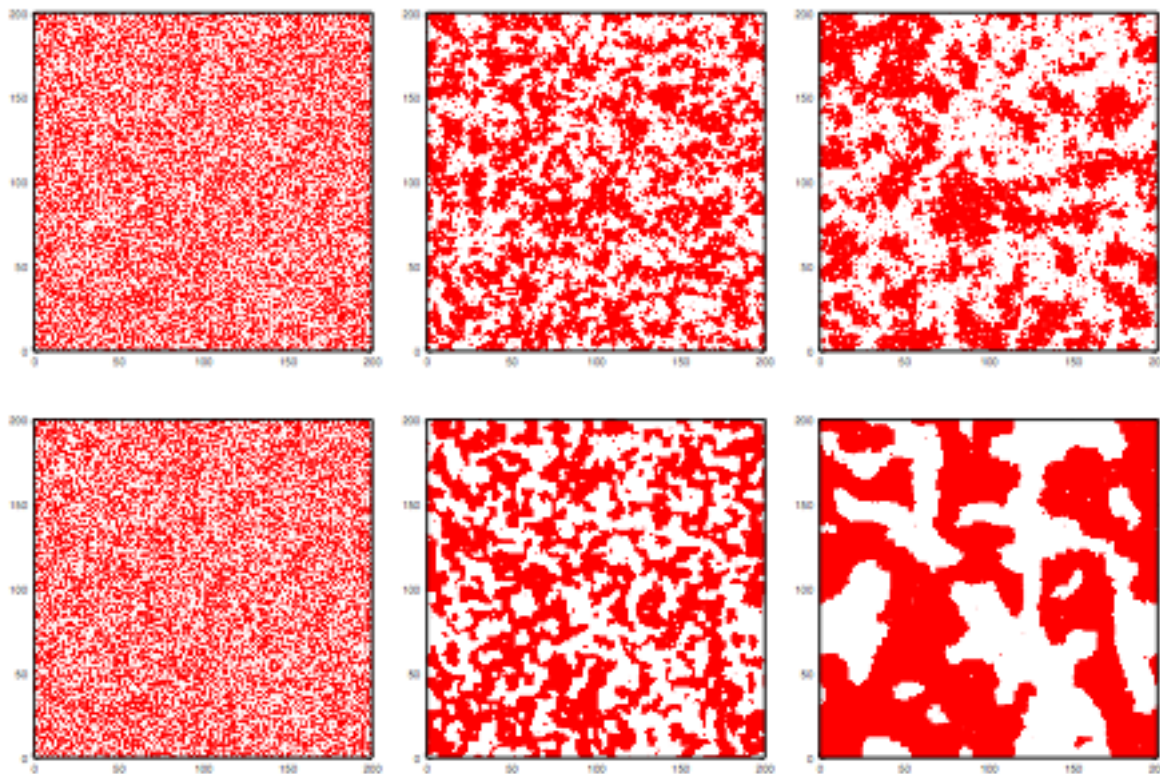
$$\mathcal{E}_{syst}(t) \neq ct, \text{ and } e_0 \ll \mathcal{E}_{syst} \ll \mathcal{E}_{env}.$$

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# $2d$ Ising model

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Snapshots after an instantaneous quench to  $T$  at  $t = 0$



$T = T_c$

$T < T_c$

At  $T = T_c$  **critical dynamics**

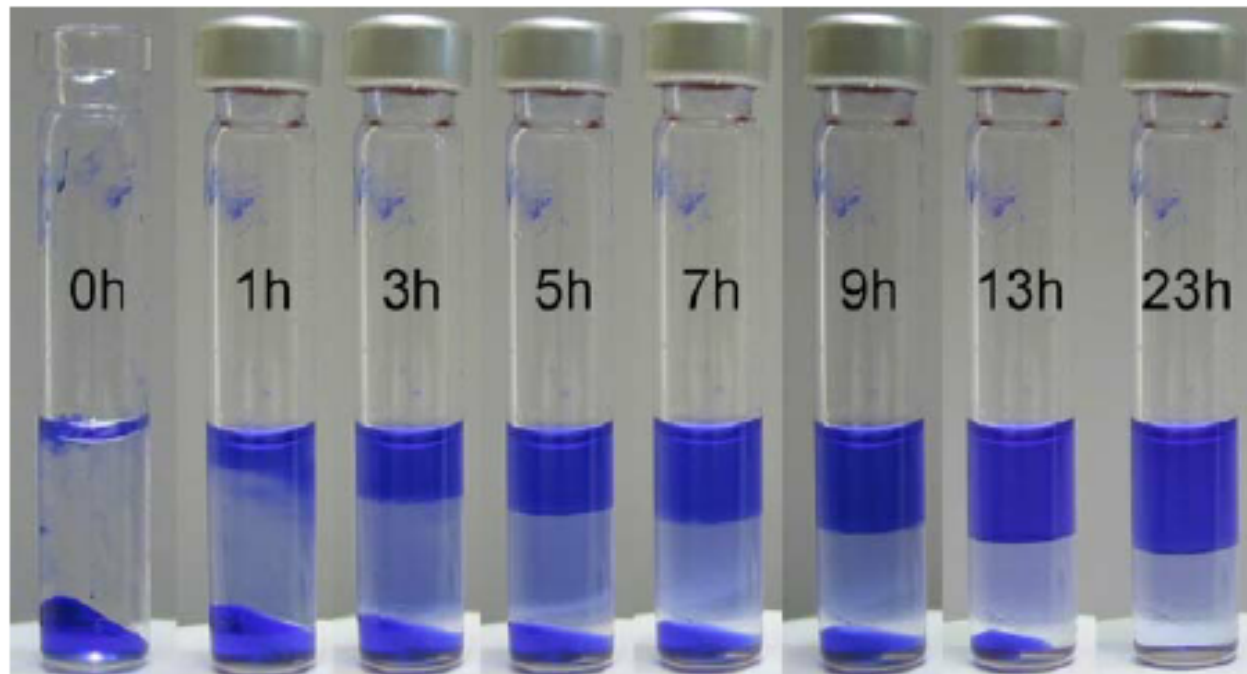
At  $T < T_c$  **coarsening**

A certain number of **interfaces** or **domain walls** in the last snapshots.

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# Membranes Proteins

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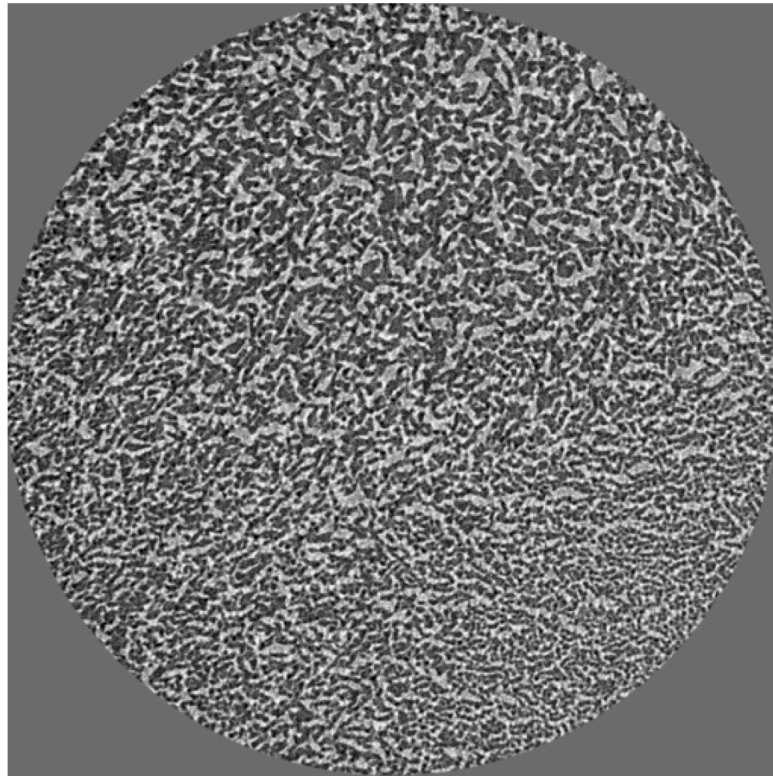
**Wadsten, Wöhri, Snijder, Katona, Gardiner, Cogdell, Neutze, Engström,**  
*Lipidic Sponge Phase Crystallization of Membrane Proteins, J. Mol. Biol. 06*



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# Phase separation in glasses

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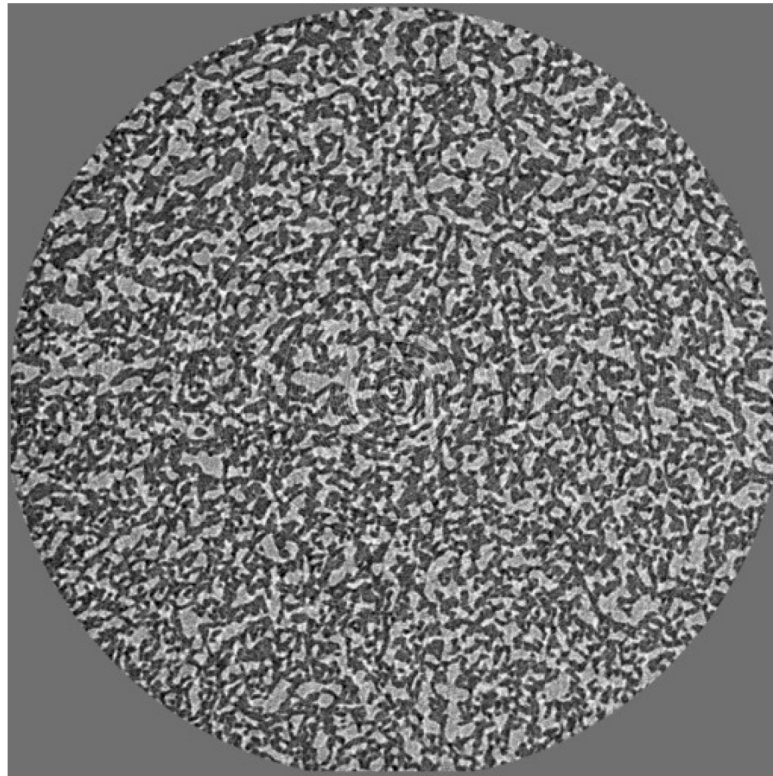
$t = 1 \text{ min}$

Gouillart (Saint-Gobain), Bouttes & Vandembroucq (ESPCI) 11-14

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# Phase separation in glasses

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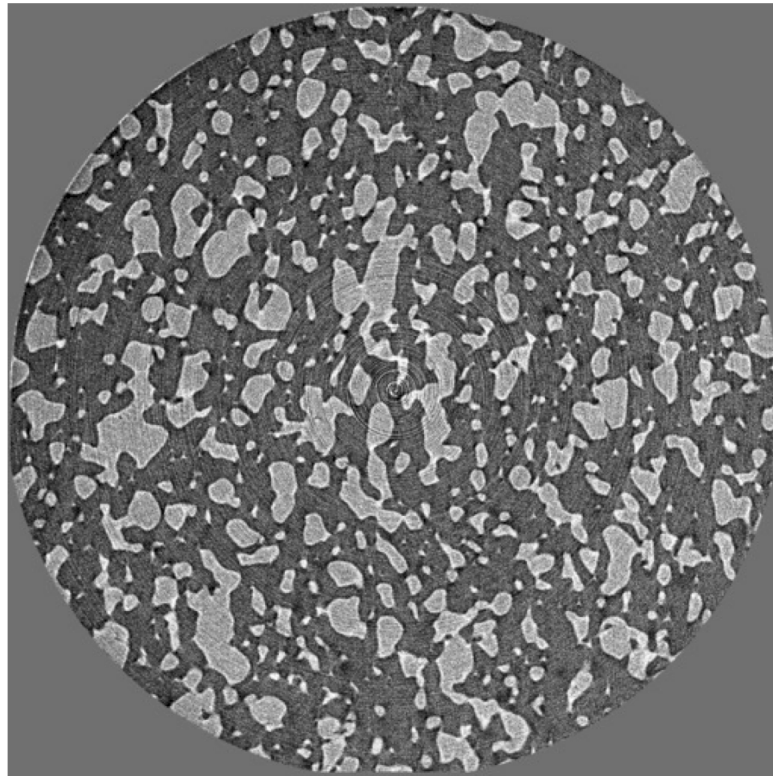
$t = 4 \text{ min}$

Gouillart (Saint-Gobain), Bouttes & Vandembroucq (ESPCI) 11-14

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# Phase separation in glasses

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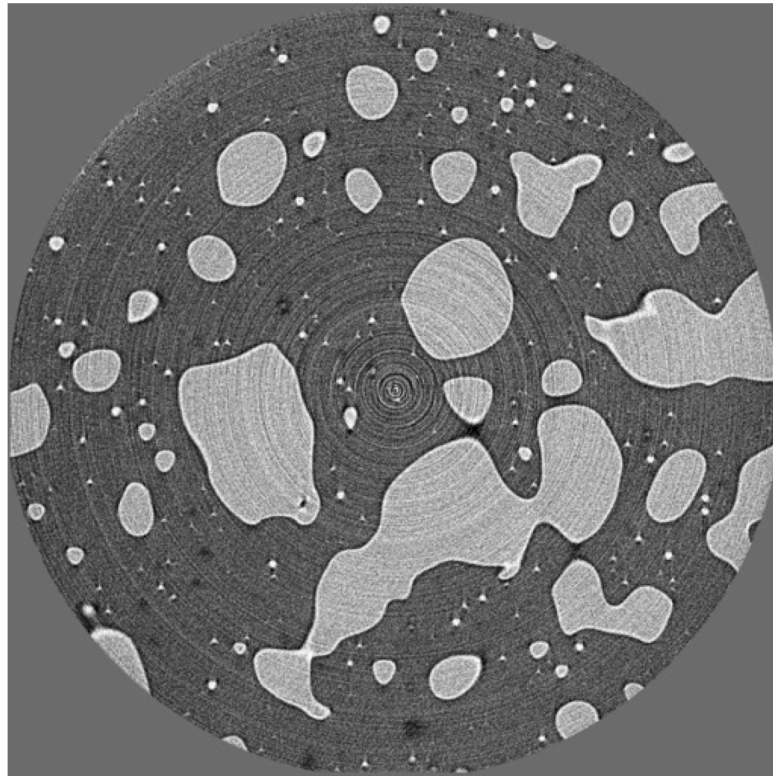
$t = 16 \text{ min}$

Gouillart (Saint-Gobain), Bouttes & Vandembroucq (ESPCI) 11-14

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# Phase separation in glasses

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$t = 64 \text{ min}$

Gouillart (Saint-Gobain), Bouttes & Vandembroucq (ESPCI) 11-14

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# $2d$ Ising Model (IM)

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Archetypical example for classical magnetic systems

The Hamiltonian or energy function is

$$H = -J \sum_{\langle ij \rangle} s_i s_j$$

$s_i = \pm 1$  Ising spins.

$\langle ij \rangle$  sum over nearest-neighbours on the lattice.

$J > 0$  ferromagnetic coupling constant.

critical temperature  $T_c > 0$  for  $d > 1$ .

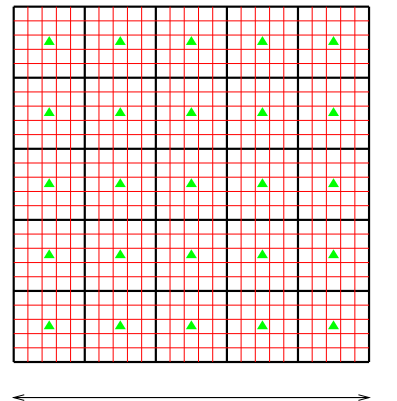
**Equilibrium** Paramagnetic & ferromagnetic states above & below  $T_c$

# Ginzburg-Landau

## Continuous scalar statistical field theory

Coarse-grain the spin

$$\phi(\vec{r}) = V_{\vec{r}}^{-1} \sum_{i \in V_{\vec{r}}} s_i.$$



The partition function is  $\mathcal{Z} = \int \mathcal{D}\phi e^{-\beta\mathcal{F}(\phi)}$  with

$$\mathcal{F}(\phi) = \int d^d r \left\{ \frac{1}{2} [\nabla \phi(\vec{r})]^2 + \frac{T-J}{2} \phi^2(\vec{r}) + \frac{g}{4} \phi^4(\vec{r}) \right\}$$

Elastic + potential energy with the latter inspired by the results for the fully-connected model (entropy around  $\phi \sim 0$  and symmetry arguments).

**Uniform** saddle point in the  $V \rightarrow \infty$  limit :  $\phi_{sp}(\vec{r}) = \langle \phi(\vec{r}) \rangle = \phi_0$ .

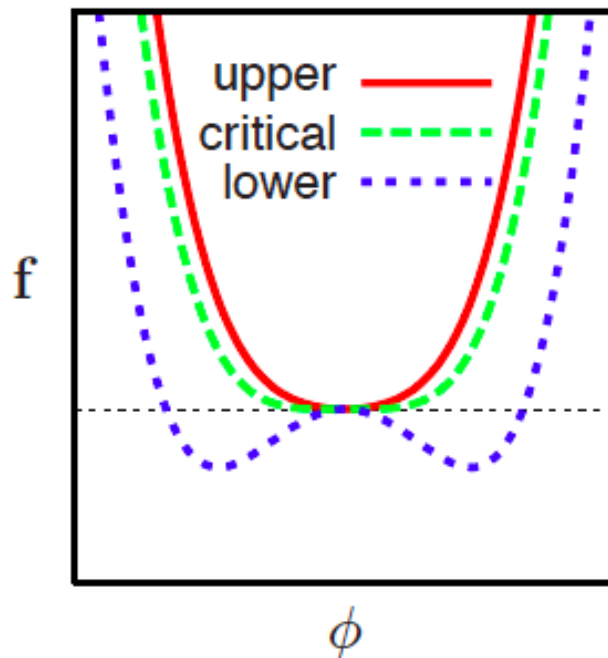
The free-energy density is  $\lim_{V \rightarrow \infty} f_V(\beta, J, g) = \lim_{V \rightarrow \infty} V^{-1} \mathcal{F}(\phi_0)$ .



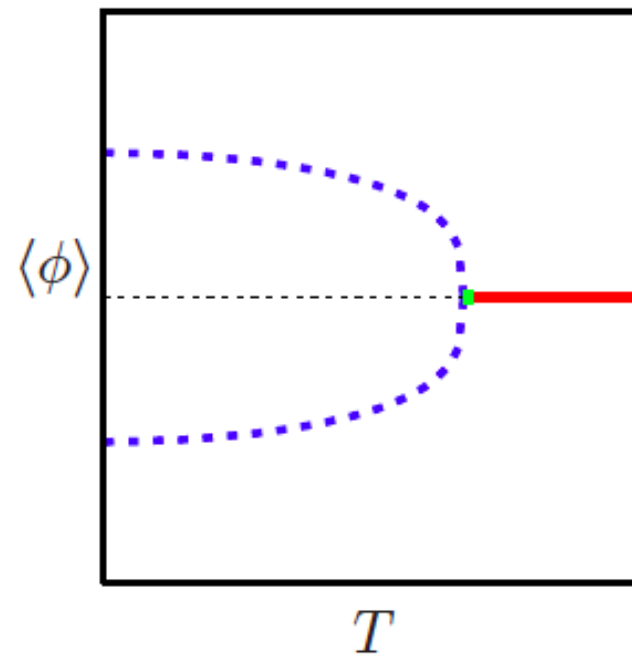
# 2nd order phase-transition

Bi-valued equilibrium states related by symmetry

$\mathcal{F}$



Ginzburg-Landau free-energy



Scalar order parameter

e.g. Ising magnets



# 2d Ising Model, dynamics

Archetypical example for classical magnetic systems

$$H = -J \sum_{\langle ij \rangle} s_i s_j$$

$s_i = \pm 1$  Ising spins.

$\langle ij \rangle$  sum over nearest-neighbours on the lattice.

$J > 0$  ferromagnetic coupling constant.

critical temperature  $T_c > 0$  for  $d > 1$ .

Monte Carlo rule  $s_i \rightarrow -s_i$  accepted with

$p = 1$	if	$\Delta\mathcal{E} < 0$
$p = e^{-\beta\Delta\mathcal{E}}$	if	$\Delta\mathcal{E} > 0$
$p = 1/2$	if	$\Delta\mathcal{E} = 0$

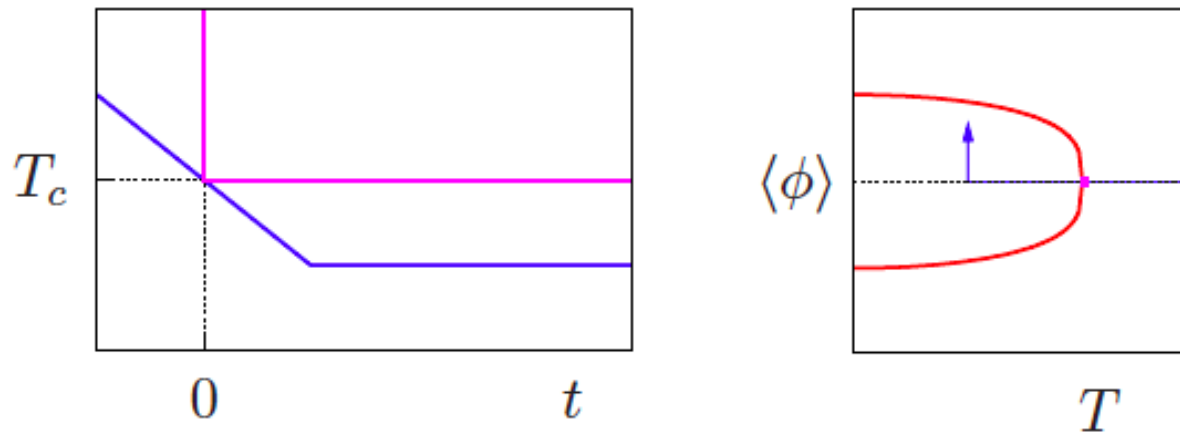
Non-conserved order parameter dynamics [  $\uparrow\downarrow$  towards  $\uparrow\uparrow$  ] etc. allowed.

[  $m = 0$  to  $m = 2$  ]

# Evolution

## Non-conserved order parameter dynamics

$$T(t) = T_c(1 - t/\tau_a)$$



Non-conserved order parameter  $\langle \phi \rangle(t, T) \neq ct$

e.g. single spin flips with Glauber or Monte Carlo stochastic rules.

**Development of magnetization in a ferromagnet.**

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# $d$ -dimensional magnets

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## More general ferromagnetic models

$$H = - \sum_{\langle ij \rangle} J_{ij} \vec{s}_i \cdot \vec{s}_j$$

$$J_{ij} > 0$$

$$\sum_{\langle ij \rangle}$$

$$s_i = \pm 1$$

$$\vec{s}_i = (s_i^x, s_i^y)$$

$$\ell^d \vec{\phi}(\vec{r}) = \sum_{i \in V_{\vec{r}}} \vec{s}_i$$

$$L$$

$$T_c > 0$$

From a pdf with positive support

Sum over nearest-neighbours on a  $d$ -dim. lattice.

Ising spins.

xy two-component spins.

Coarse-grained field over the volume  $V = \ell^d$

Linear size of the system  $L \gg \ell \gg a$

for  $d > 1$  and  $L \rightarrow \infty$ .

Coupling to the bath mimicked by **Monte Carlo updates**

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# Stochastic dynamics

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## Open systems

- **Microscopic**: identify the ‘smallest’ relevant variables in the problem (*e.g.*, the spins) and propose stochastic updates for them, as the **Monte Carlo or Glauber** rules.
- **Coarse-grained**: write down a stochastic differential equation for the field, such as the **effective (Markov) Langevin equation**

$$\underbrace{m\ddot{\vec{\phi}}(\vec{r}, t)}_{\text{Inertia}} + \underbrace{\gamma_0\dot{\vec{\phi}}(\vec{r}, t)}_{\text{Dissipation}} = \underbrace{\vec{F}(\vec{\phi})}_{\text{Deterministic}} + \underbrace{\vec{\xi}(\vec{r}, t)}_{\text{Noise}}$$

with  $\vec{F}(\vec{\phi}) = -\delta\mathcal{F}(\vec{\phi})/\delta\vec{\phi}$  (with the double-well  $\mathbf{f}$ )

*e.g.*, time-dependent stochastic scalar Ginzburg-Landau equation or the stochastic Gross-Pitaevskii equation.

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# Models

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## Discrete vs. continuous

### Ising spin models

$$H = - \sum_{ij} J_{ij} S_i S_j$$

NCOP [  $\uparrow\downarrow \mapsto \uparrow\uparrow$  ]

COP [  $\uparrow\downarrow \mapsto \downarrow\uparrow$  ]

### Field theories

$$\mathcal{F}[\phi] = \int d^d r \left[ \frac{1}{2} (\nabla \phi)^2 - \frac{\mu}{2} \phi^2 + \frac{g}{4} \phi^4 \right]$$

$$\partial_t \phi(\vec{r}, t) = \delta_{\phi(\vec{r}, t)} \mathcal{F}[\phi] + \xi(\vec{r}, t)$$

$$\partial_t \phi(\vec{r}, t) = \nabla^2 \delta_{\phi(\vec{r}, t)} \mathcal{F}[\phi] + \eta(\vec{r}, t)$$

Overdamped limit is fine

In the COP case  $\langle \eta(\vec{x}, t) \eta(\vec{y}, t') \rangle = 2k_B T \nabla^2 \delta(\vec{x} - \vec{y}) \delta(t - t')$

And generalisations for vector cases. **Quenched disorder** can be introduced by taking the  $J_{ij}$  or the parameters in the field theory from a pdf.

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# Plan of the lecture

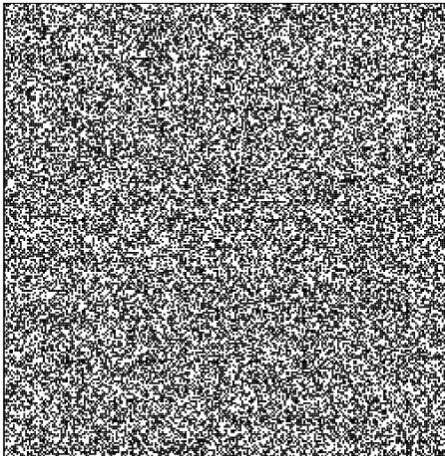
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# Equilibrium configurations

Up & down spins in a  $2d$  Ising model

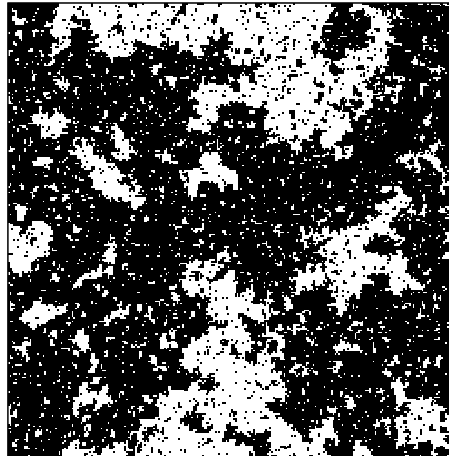
$$T \rightarrow \infty$$



$$\langle s_i \rangle_{eq} = 0$$

$$\phi(\vec{r}) = 0$$

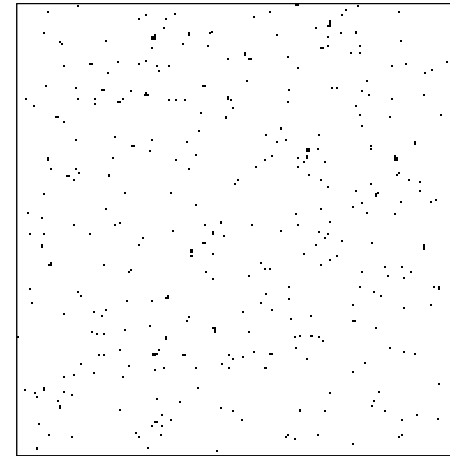
$$T = T_c$$



$$\langle s_i \rangle_{eq} = 0$$

$$\phi(\vec{r}) = 0$$

$$T < T_c$$



$$\langle s_i \rangle_{eq+} > 0$$

$$\phi(\vec{r}) > 0$$

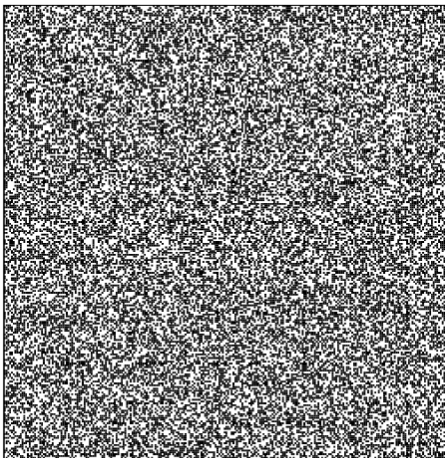
Coarse-grained scalar field  $\phi(\vec{r}) \equiv \frac{1}{V_{\vec{r}}} \sum_{i \in V_{\vec{r}}} s_i$

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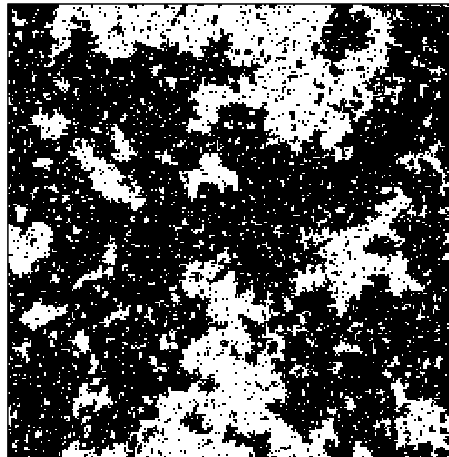
# The problem

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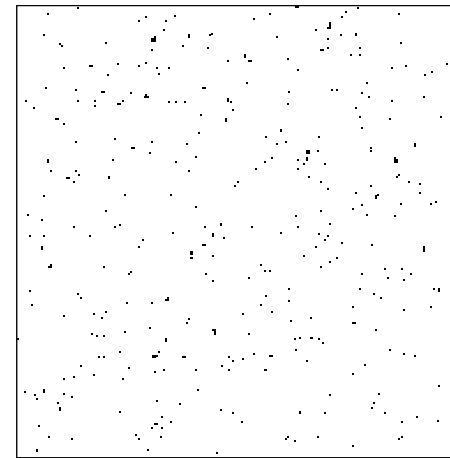
Up & down spins in a  $2d$  Ising model



$$T \rightarrow \infty$$



$$T = T_c$$



$$T < T_c$$

**Question** : starting from equilibrium at  $T_0 \rightarrow \infty$  or  $T_0 = T_c$  how is equilibrium at  $T = T_c$  or  $T < T_c$  attained ?

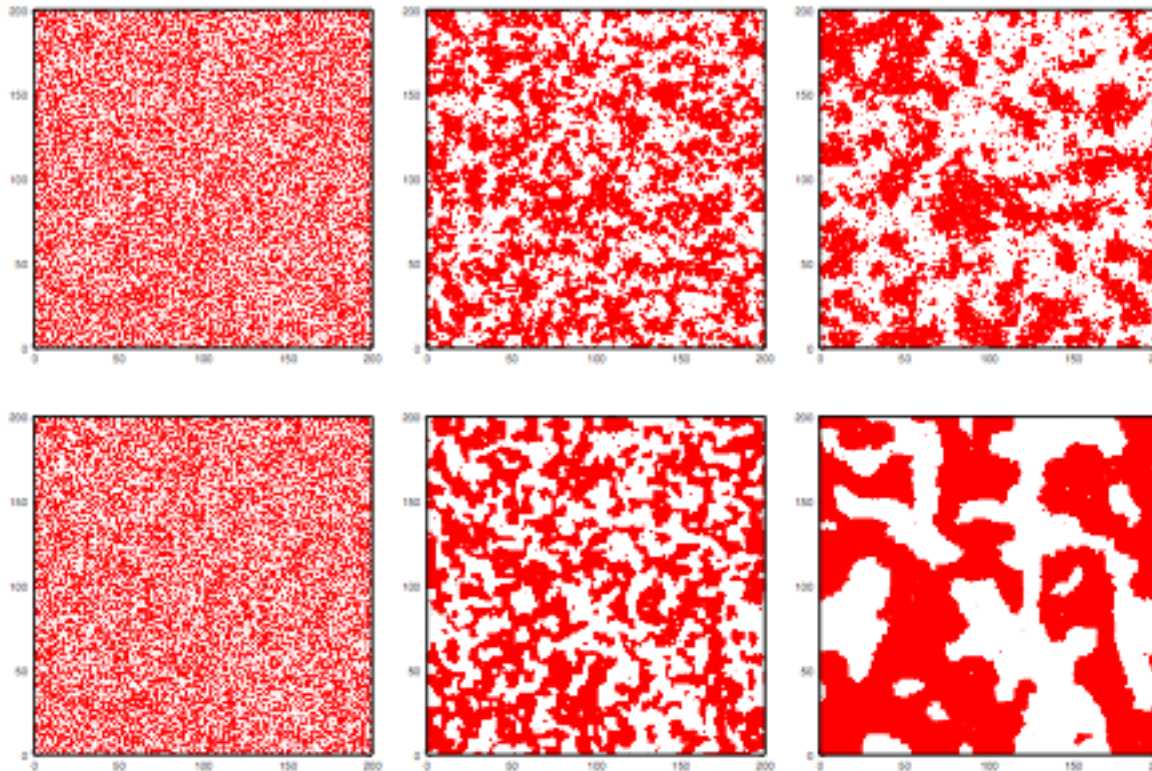


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# $2d$ Ising model

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Snapshots after an instantaneous quench at  $t = 0$



$T = T_c$

$T < T_c$

At  $T = T_c$  **critical dynamics**

At  $T < T_c$  **coarsening**

A certain number of **interfaces** or **domain walls** in the last snapshots.

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# Domain growth

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- At  $T = T_c$  the system needs to grow structures of all sizes.

## Critical coarsening.

- At  $T < T_c$  : the system tries to order locally in one of the two competing equilibrium states at the new conditions.

## Sub-critical coarsening.

The **linear size of the equilibrated patches increases in time.**

- The relaxation time  $t_r$  needed to reach  $\langle \phi \rangle_{eq}(T/J)$  diverges with the size of the system,  $t_r(T/J, L) \rightarrow \infty$  when  $L \rightarrow \infty$  for  $T \leq T_c$ .
- Dissipative dynamics  $d\langle \mathcal{E} \rangle / dt < 0$ . Energy density is reduced by diminishing the density of **domain walls**.

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# Statement

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In both cases one sees the growth of 'red and white' patches and **interfaces** surrounding such geometric domains.

More precisely, spatial regions of local equilibrium (with vanishing or non-vanishing order parameter) grow in time and

a **growing length**  $\mathcal{R}(t, T/J)$  can be computed with the help of dynamic scaling.

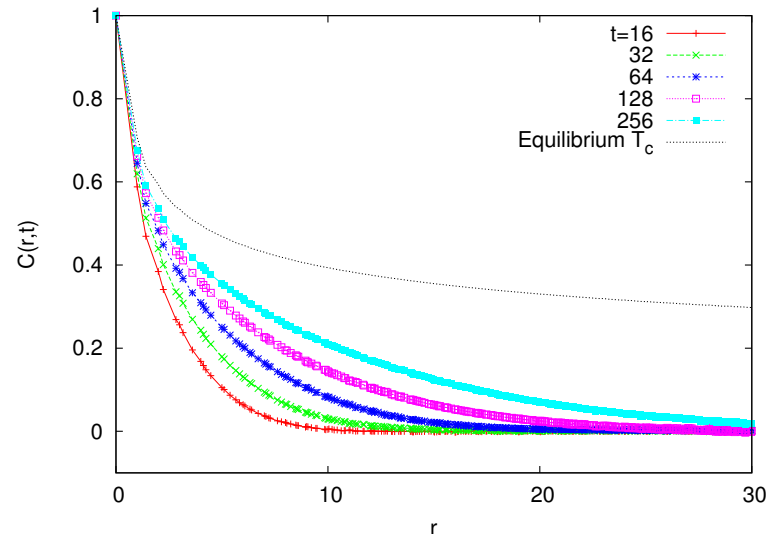
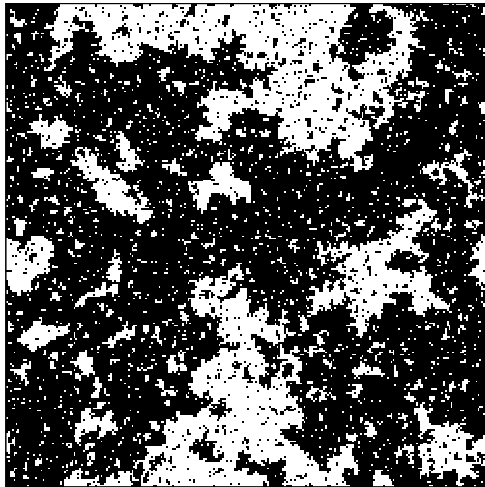
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# Critical growth



- Black curve : equilibrium relaxation,  $r^{2-d-\eta}$ .
- Coloured curves are for different times after the quench and they slowly approach the equilibrium one.
- From  $C(\mathcal{R}_c(t), t) = 1/e$  one gets  $\mathcal{R}_c(t) \simeq t^{1/z_{eq}}$   
(Other prescriptions give equivalent results.)

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# Dynamic scaling

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After quenches (set  $J = 1$  for notational simplicity)

At late times there is a single *length-scale*, the *typical radius of the equilibrium structures (domains below  $T_c$ )*  $\mathcal{R}(t, T)$ , such that the structure is (in statistical sense) independent of time when lengths are scaled by  $\mathcal{R}(t, T)$ , e.g.

$$C(r, t) \equiv \langle s_i(t) s_j(t) \rangle_{|\vec{x}_i - \vec{x}_j| = r} \sim f \left( \frac{r}{\mathcal{R}(t, T)} \right),$$

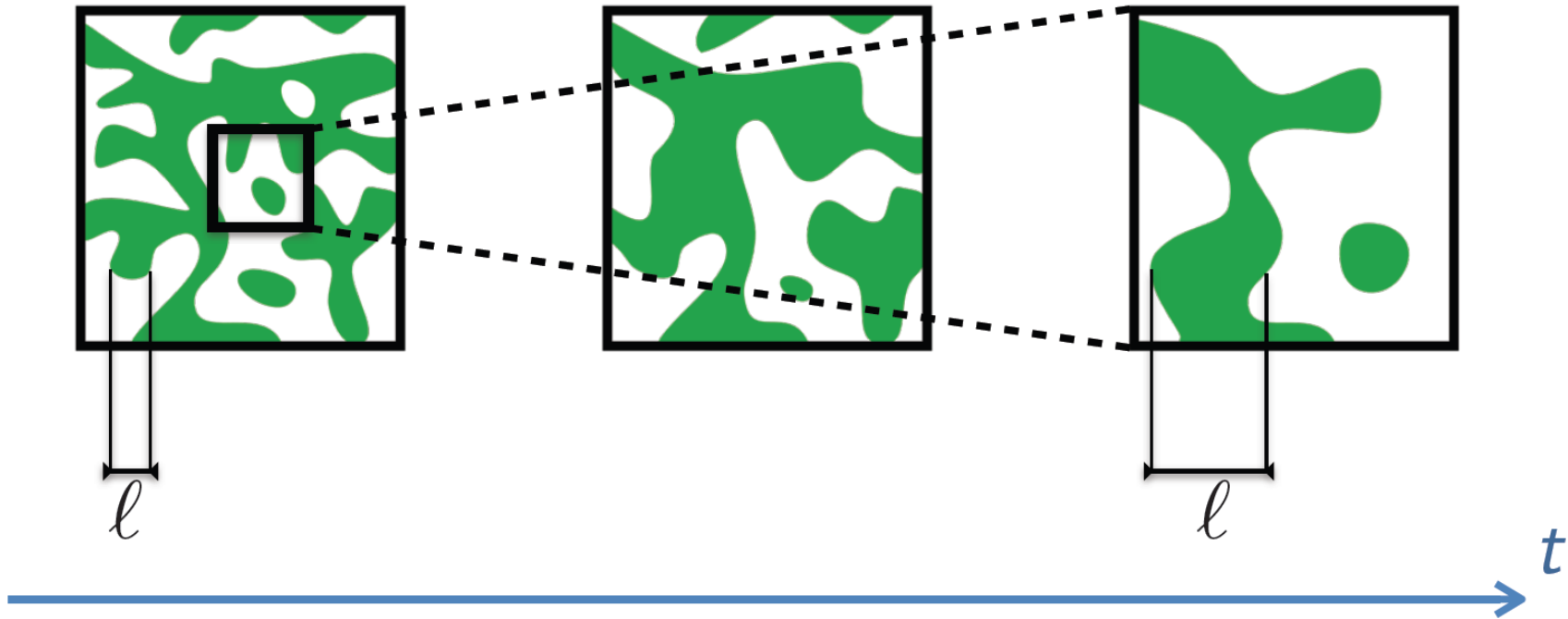
$$C(t, t_w) \equiv \langle s_i(t) s_i(t_w) \rangle \sim f_c \left( \frac{\mathcal{R}(t, T)}{\mathcal{R}(t_w, T)} \right),$$

etc. when  $L \gg r \gg \xi(T)$ ,  $t, t_w \gg t_0$  and  $C$  small enough (see below).

Suggested by experiments and numerical simulations. Proved for a few cases.

# Dynamic scaling

in phase ordering kinetics



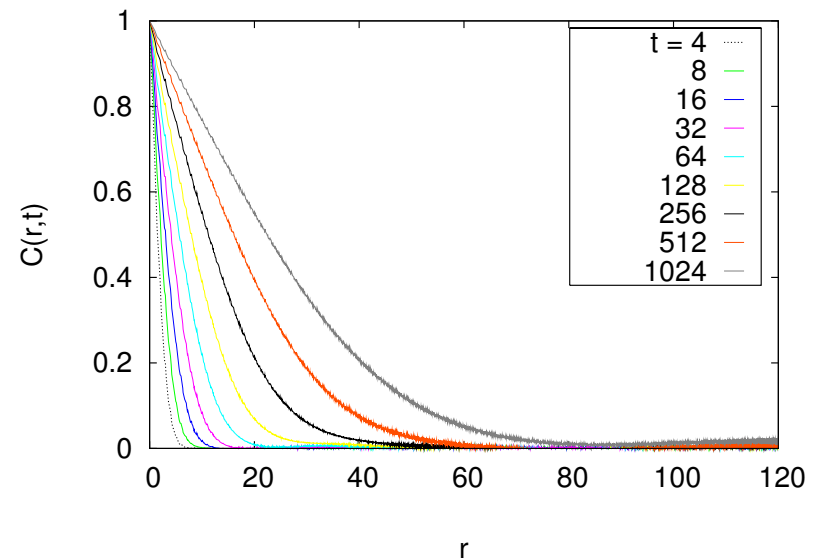
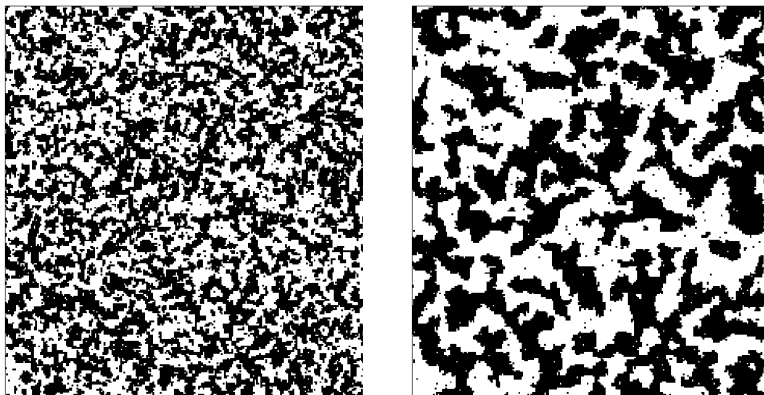
Growing length  $\ell(t)$  and equilibrium reached for  $\ell(t_{eq}) \simeq L$

Typically  $\ell(t) \simeq t^{1/z}$  and  $t_{eq} \simeq L^z$

Excess energy w.r.t. the equilibrium one stored in the domain walls;  $\Delta\mathcal{E}(t) \simeq \ell^{-a}(t)$

# Dynamic scaling

Quench of the  $2d$ IM with NCOP from  $T_0 \rightarrow \infty$  to  $T = 0$



$$\langle \phi(t) \rangle = 0 \quad C_{eq}^c(r) \simeq e^{-r/\xi_{eq}}$$

- Coloured curves are  $C(r, t)$  for different times after the quench.
- The growing length is  $\mathcal{R}(t, T) \simeq t^{1/z_d}$  with  $z_d = 2$
- $\mathcal{R}(t, T)$  is the averaged linear size of the domains.



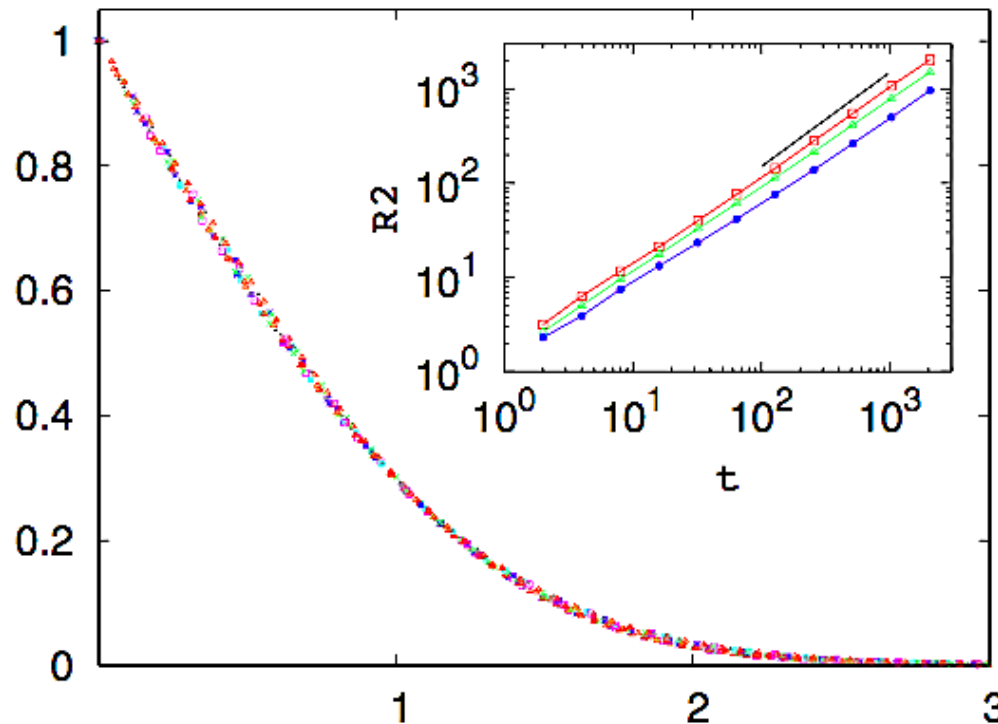
# Dynamic scaling

Quench of the  $2d$ IM with NCOP from  $T_0 \rightarrow \infty$  to  $T < T_c$

Scaling regime  $a \ll r \ll L$ ,  $r \simeq \mathcal{R}(t, T) \simeq t^{1/z_d}$

$$C(r, t) \simeq m_{eq}^2(T) f_c \left( \frac{r}{\mathcal{R}(t, T)} \right)$$

Scaling looks perfect



$r/\mathcal{R}(t, T)$

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# Space-time correlation

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## Separation of time-scales & dynamic scaling

### Critical quench

$$C(r, t) \simeq C_{eq}(r) f_c \left( \frac{r}{\mathcal{R}_c(t)} \right)$$

$$C_{eq}(r) \simeq r^{2-d-\eta}, \lim_{x \rightarrow 0} f_c(x) = 1 \text{ and } \lim_{x \rightarrow \infty} f_c(x) = 0.$$

### Sub-critical

$$C(r, t) \simeq [C_{eq}(r) - m_{eq}^2] + m_{eq}^2 f \left( \frac{r}{\mathcal{R}(t, T)} \right)$$

$$C(0, t) = 1 \quad \forall t, \lim_{r \rightarrow 0} C_{eq}(r) = 1, \lim_{r \rightarrow \infty} C_{eq}(r) \propto \langle s_i \rangle_{eq}^2 = m_{eq}^2,$$

$\lim_{x \rightarrow 0} f(x) = 1$  (long times) and  $\lim_{x \rightarrow \infty} f(x) = 0$  (short distances).

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# Growing length

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## Dynamic universality classes

Use the growing length  $\mathcal{R}(t, T)$  to identify dynamic universality classes.

They depend on the dimension of the order parameter and the dynamic mechanism of growth (intimately related to the conservation laws).

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# Growing length

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## Dynamic universality classes at the critical point

At  $T_c$ , dynamic RG techniques work very well.

**U. C. Tauber**, *Critical Dynamics : A Field Theory Approach to Equilibrium and Non-Equilibrium Scaling Behavior* (Cambridge University Press, 2014)

One finds dynamic scaling, with the growing length

$$\mathcal{R}_c(t) \simeq t^{1/z_c}$$

$z_c$  can be computed with methods that are very similar to critical exponents in static phase transitions.

Dynamic universality classes classified by the  $z_c$  values.

The scaling functions can be estimated as well

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# Growing length

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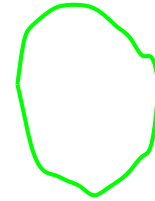
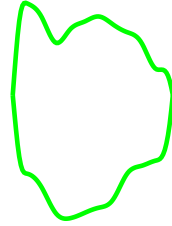
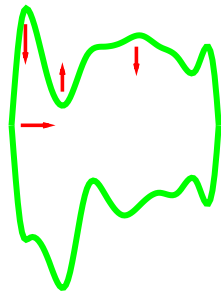
Dynamic universality classes below the critical point

No systematic method

Focus on the dynamic mechanisms

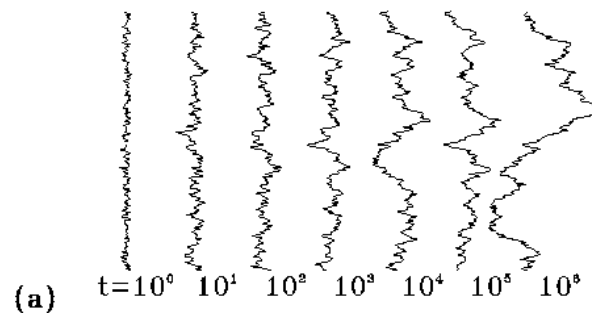
# Scalar field w/NCOP dynamics

- Curvature driven ( $T = 0$ ):  $\vec{v} \equiv \frac{d\vec{n}}{dt} \propto K \hat{n}$  with  $K = \vec{\nabla} \cdot \hat{n}$



Allen & Cahn 79

- Domain wall roughening ( $T > 0$ )
- Domain wall roughening and pinning by quenched disorder



e.g. elastic line in random media. Kolton *et al* 05

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# Curvature driven

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Numerical solution of the time-dependent Ginzburg-Landau equation

**A. Langins (1st year master project)**



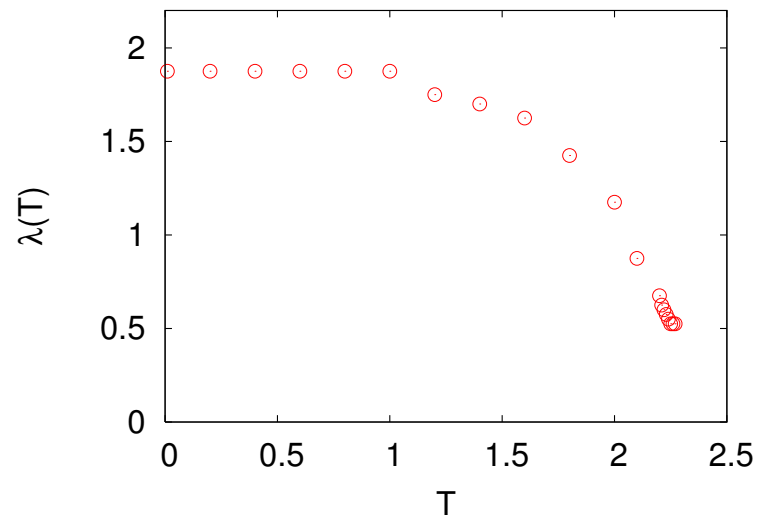
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# MC dynamics 2dIM

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The typical length-scale  $\Leftrightarrow$  a typical area

$$\mathcal{R}(t, T) \sim \lambda(T) t^{1/2} \quad \Leftrightarrow \quad \mathcal{A}(t, T) \sim \lambda^2(T) t$$



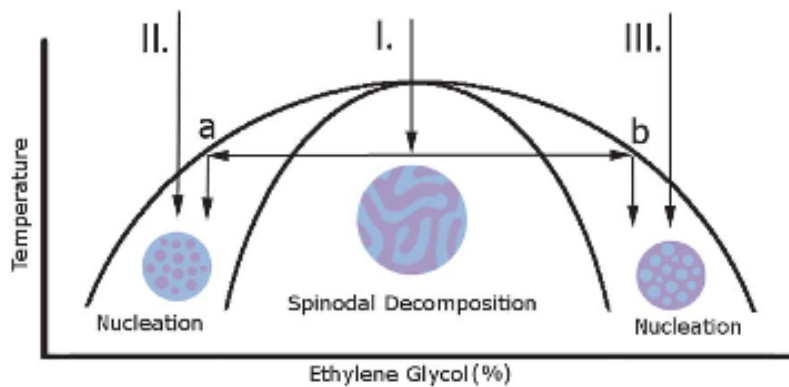
NB the exponent  $\frac{1}{2}$  is independent of  $T$  and the details of the dynamics, lattice, *etc.* as long as the order parameter is non-conserved & there is no disorder.

The  $T$ -dependence in  $\lambda(T)$  is due to the roughening of the domain walls.

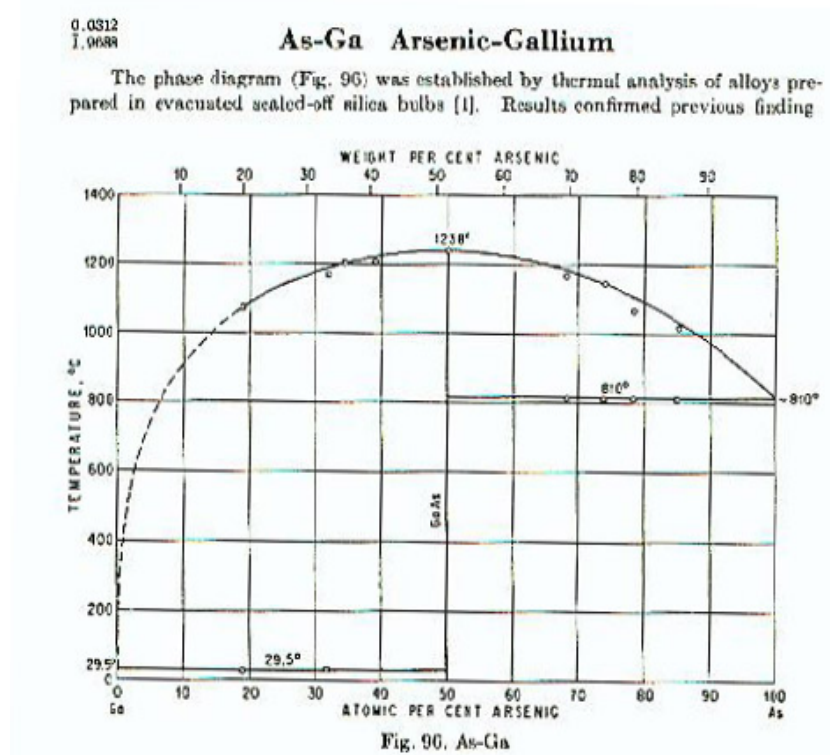
# Phase separation

## Demixing transitions

Two species ● and ●, repulsive interactions between them.



Sketch



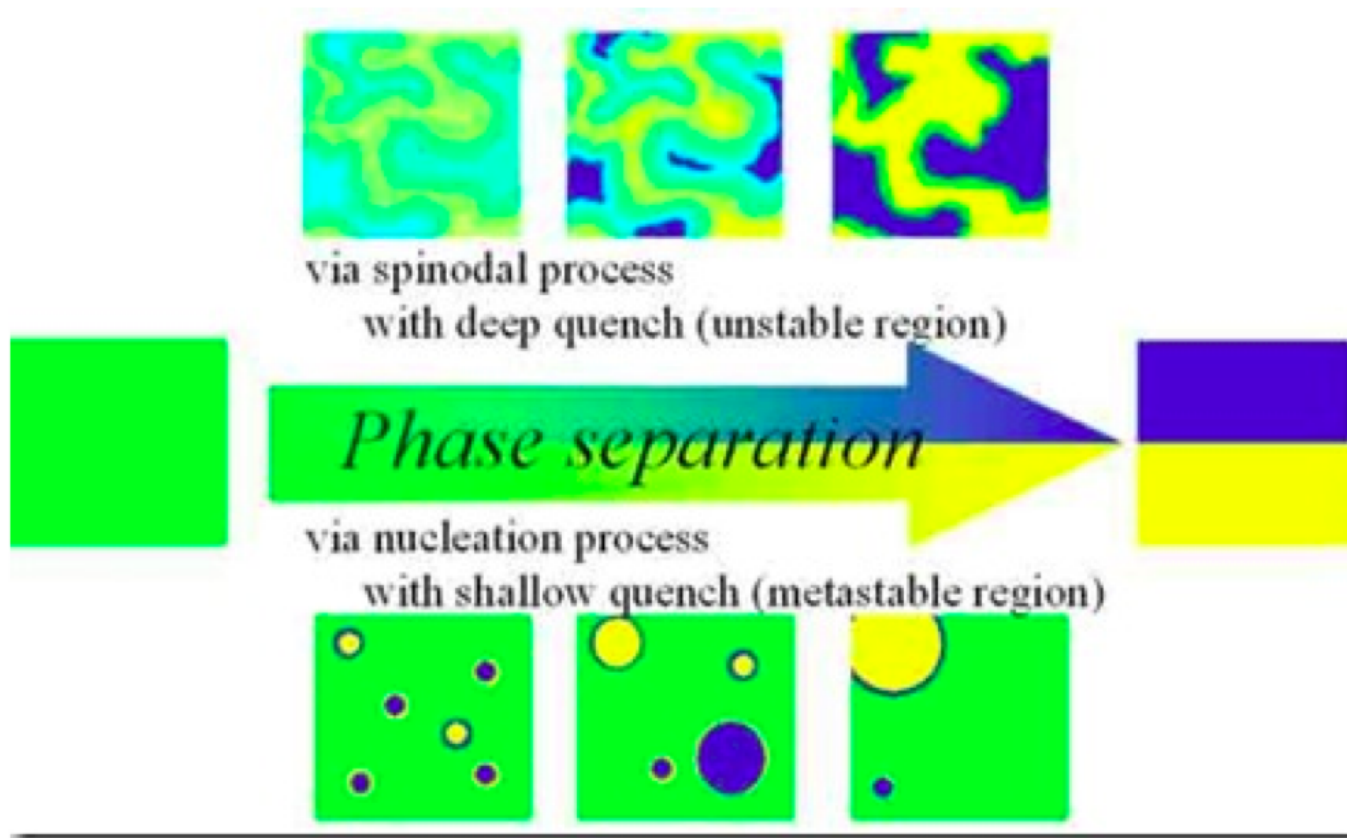
Experimental phase diagram

Binary alloy, **Hansen & Anderko, 54**

# Scalar field w/COP

## Phase separation

Matter diffusion



---

# Phase separation

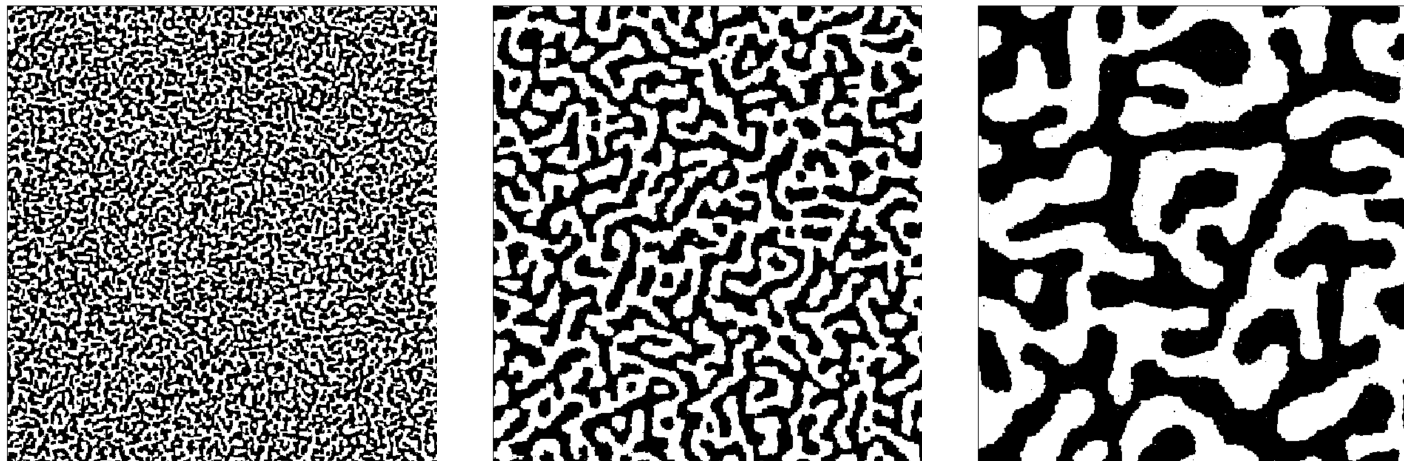
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## Spinodal decomposition in binary mixtures

$A$  species  $\equiv$  spin up;  $B$  species  $\equiv$  spin down

$2d$  Ising model with Kawasaki dynamics at  $T$

locally conserved order parameter



50 : 50 composition Rounder boundaries

$$\mathcal{R}(t, T) \simeq \lambda(T)t^{1/3}$$

Huse 93

---

# Weak disorder

---

e.g., random ferromagnets

At short time scales the dynamics is relatively fast and independent of the quenched disorder ; thus

$$\mathcal{R}(t, T) \simeq \lambda(T) t^{1/z_d}$$

At longer time scales **domain-wall pinning** by disorder dominates.

Assume there is a length-dependent **barrier**  $B(\mathcal{R}) \simeq \Upsilon \mathcal{R}^\psi$  to overcome

The **Arrhenius time** needed to go over such a barrier is  $t_A \simeq t_0 e^{\frac{B(\mathcal{R})}{k_B T}}$

This implies

$$\mathcal{R}(t, T) \simeq \left( \frac{k_B T}{\Upsilon} \ln t/t_0 \right)^{1/\psi}$$

---

# Weak disorder

---

Still two ferromagnetic states related by symmetry

$$\mathcal{R}(t, T) \simeq \begin{cases} \lambda(T)t^{1/z_d} & \mathcal{R} \ll L_c(T) \quad \text{curvature-driven} \\ L_c(T)(\ln t/t_0)^{1/\psi} & \mathcal{R} \gg L_c(T) \quad \text{activated} \end{cases}$$

with  $L_c(T)$  a growing function of  $T$ .

Inverting times as a function of length  $t \simeq [\mathcal{R}/\lambda(T)]^{z_d} e^{\mathcal{R}/L_c(T)}$

At short times this equation can be approximated by an effective power law with a  $T$ -dependent exponent :

$$t \simeq \mathcal{R}^{\bar{z}_d(T)} \quad \bar{z}_d(T) \simeq z_d [1 + ct/L_c(T)]$$

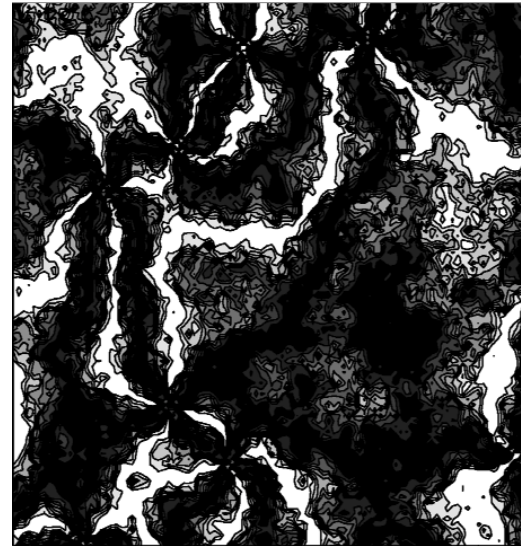
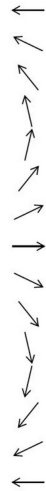
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# Planar magnets

---

**Schrielen pattern** : gray scale according to  $\sin^2 2\theta_i(t)$

Spin-waves      Vortices (planar spins turn around these points)



After a quench vortices annihilate and tend to bind in pairs

$$\mathcal{R}(t, T) \simeq \lambda(T) [t / \ln(t/t_0(T))]^{1/2}$$

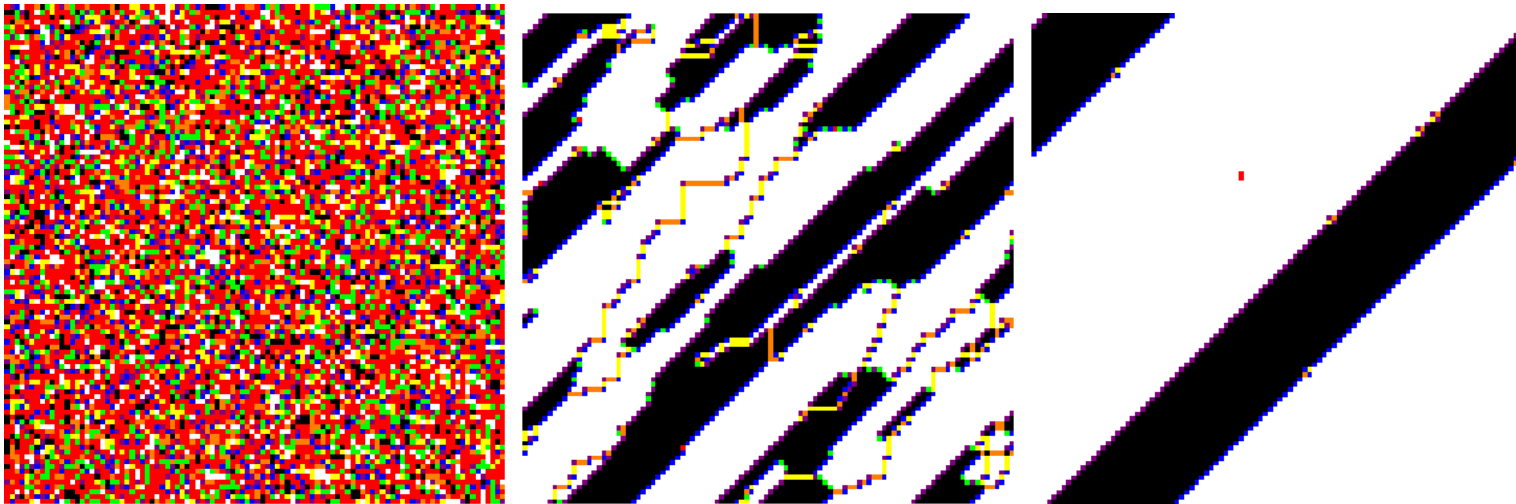
**Yurke *et al* 93, Bray & Rutenberg 94, Jelic & LFC 11**

---

# Frustrated magnets

---

e.g.,  $2d$  spin ice or vertex models



Stripe growth in the FM phase

Anisotropic growth,  $\mathcal{R}_\perp(t, T)$  and  $\mathcal{R}_\parallel(t, T)$



---

# Universality classes

---

as classified by the growing length

$$\mathcal{R}(t, T) \simeq \left\{ \begin{array}{ll} \lambda(T) t^{1/2} & \text{scalar NCOP} \quad z_d = 2 \\ \lambda(T) t^{1/3} & \text{scalar COP} \quad z_d = 3 \\ \lambda(T) \left( \frac{t}{\ln t} \right)^{1/2} & \text{planar NCOP in} \quad d = 2 \\ \text{etc.} & \end{array} \right.$$

Temperature and other microscopic parameters appear in the prefactor.

---

# Universality classes

---

as classified by the growing length

$$\mathcal{R}(t, T) \simeq \begin{cases} \lambda(T) t^{1/2} & \text{scalar NCOP} & z_d = 2 \\ \lambda(T) t^{1/3} & \text{scalar COP} & z_d = 3 \\ \lambda(T) \left( \frac{t}{\ln t} \right)^{1/2} & \text{planar NCOP in } d = 2 \\ \lambda(T) (\ln t)^{1/\psi} & \text{weak disorder NCOP} \end{cases}$$

Are scaling functions independent of  
temperature, other parameters, microscopic dynamics ?

Super-universality ?

---

# Plan of the lecture

---

1. The phenomenon
2. Theoretical setting
3. Critical and sub-critical quenches
4. **Dynamic scaling (again)**
5. Dynamic universality classes
6. Two-time correlations and ageing
7. Two-time responses and loss of memory
8. Mean-field models
9. Modern studies

---

# Dynamic scaling

---

## Scaling functions

very early MC simulations **Lebowitz et al 70s** & experiments

One identifies a **growing linear size of equilibrated patches**

$$\mathcal{R}(t, T)$$

If this is the **only** length governing the dynamics, the **space-time correlation functions** should scale with  $\mathcal{R}(t, T)$  according to

At  $T = T_c$        $C(r, t) \simeq C_{eq}(r) f_c\left(\frac{r}{\mathcal{R}_c(t)}\right)$       **Scaling fct  $f_c$  ✓**

At  $T < T_c$        $C(r, t) \simeq C_{eq}^c(r) + m_{eq}^2 f\left(\frac{r}{\mathcal{R}(t, T)}\right)$       **Scaling fct  $f$  ?**

Reviews **Hohenberg & Halperin 77** (critical) **Bray 94** (sub-critical)

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# Plan of the lecture

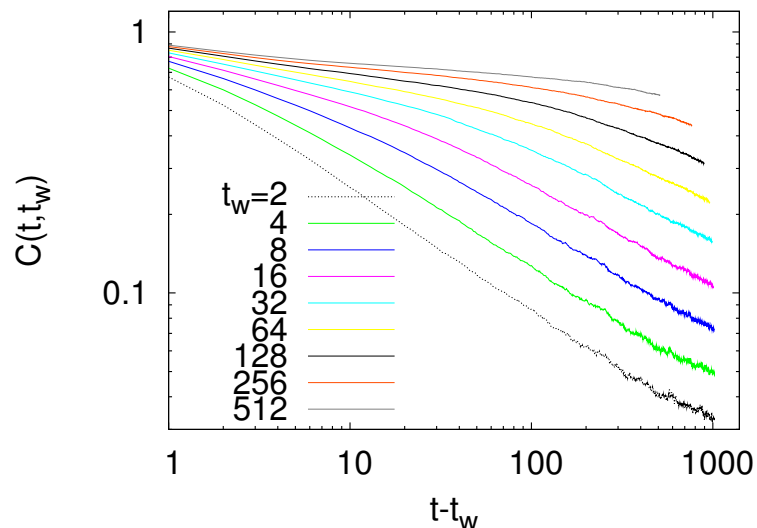
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1. The phenomenon
2. Theoretical setting
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4. Dynamic scaling
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# Two-time self-correlation

e.g., MC simulation of the  $2dIM$  at  $T < T_c$

$$C(t, t_w) = N^{-1} \sum_{i=1}^N \langle s_i(t) s_i(t_w) \rangle$$



Stationary relaxation

Aging decay

Separation of time-scales : stationary – aging

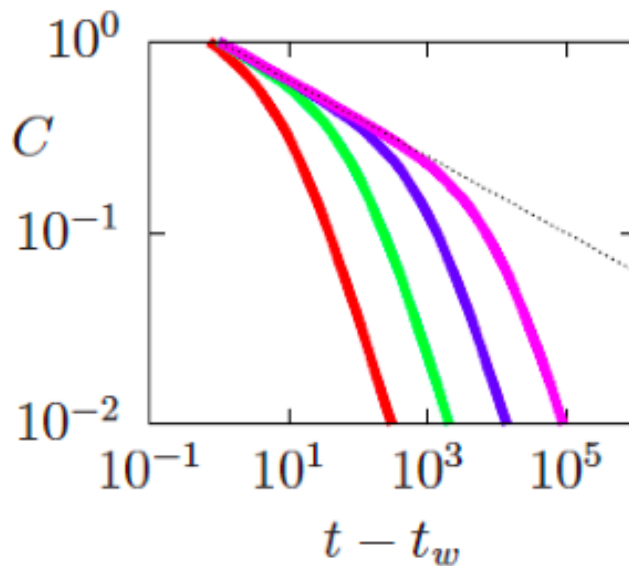
$$C(t, t_w) = C_{st}(t - t_w) + m_{eq}^2 f \left( \frac{\mathcal{R}(t, T)}{\mathcal{R}(t_w, T)} \right)$$

$$C_{st}(0) = 1 - m_{eq}^2, \lim_{x \rightarrow \infty} C_{st}(x) = 0, f(1) = 1, \lim_{x \rightarrow \infty} f(x) = 0.$$

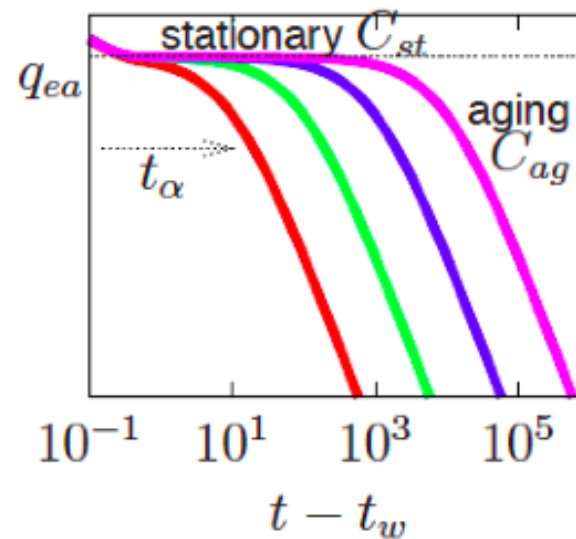
# Two-time self-correlation

## Comparison

Critical coarsening ( $T = T_c$ )



Sub-critical coarsening ( $T < T_c$ )



## Separation of time-scales

Multiplicative

Additive

---

# Aging

---

## Older samples relax more slowly

Older samples need more time to relax

spontaneously (correlation functions)

after a change in conditions (response functions)

$t_w$  is the time that measures the age of the system

Huge literature on this phenomenology. Some reviews of experimental measurements were written by **Struick** on polymer glasses, **Vincent et al.** & **Nordblad et al.** on spin-glasses, **McKenna et al.** on all kinds of glasses.



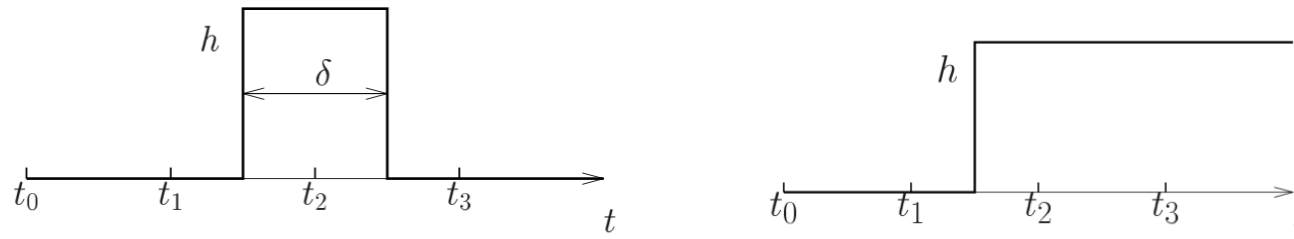
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# Plan of the lecture

---

1. The phenomenon
2. Theoretical setting
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4. Dynamic scaling
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6. Two-time correlations and ageing
7. **Two-time responses and loss of memory**
8. Mean-field models
9. Modern studies

# Response to perturbations



The **perturbation** couples **linearly** to the observable  $B[\{s_i\}]$

$$H \rightarrow H - hB[\{s_i\}]$$

The **linear instantaneous response** of another observable  $A[\{s_i\}]$  is

$$R_{AB}(t, t_w) \equiv \left\langle \frac{\delta A[\{s_i\}](t)}{\delta h(t_w)} \Big|_{h=0} \right\rangle$$

The **linear integrated response** or **dc susceptibility** is

$$\chi_{AB}(t, t_w) \equiv \int_{t_w}^t dt' R_{AB}(t, t')$$

---

# Linear response

---

## Critical and sub-critical coarsening

### Critical coarsening

$$\chi(t, t_w) = \beta - \chi_{eq}(t - t_w)g \left( \frac{\mathcal{R}(t, T)}{\mathcal{R}(t_w, T)} \right)$$

### Sub-critical coarsening

$$\chi(t, t_w) = \chi_{eq}(t - t_w) + [\mathcal{R}(t_w, T)]^{-a_x} g \left( \frac{\mathcal{R}(t, T)}{\mathcal{R}(t_w, T)} \right)$$

In both cases :  $\chi_{eq}(t - t_w) = -(k_B T)^{-1} dC_{eq}(t - t_w)/d(t - t_w)$ .

**To be proven in the 3rd Lecture**

Reviews

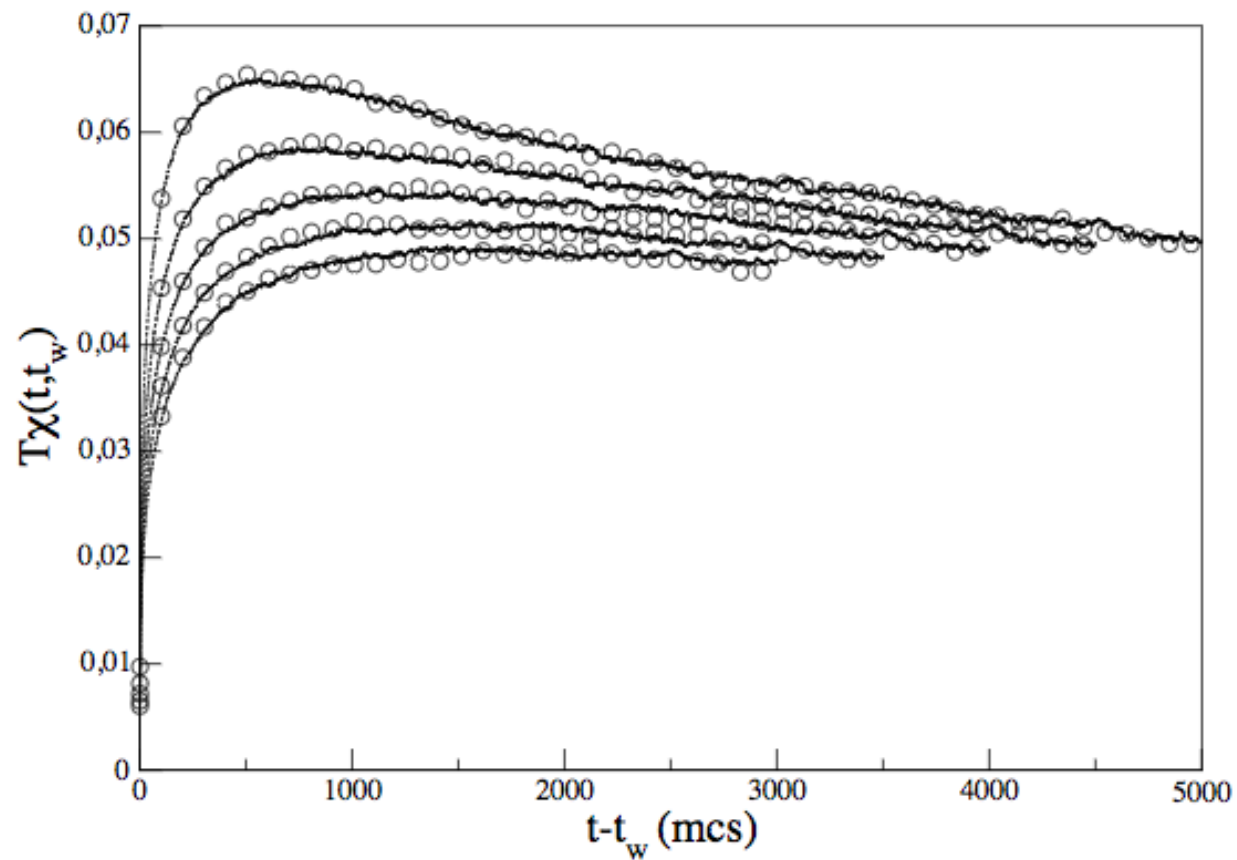
**Crisanti & Ritort 03, Calabrese & Gambassi 05, Corberi *et al.* 07, LFC 11**

---

# Linear response

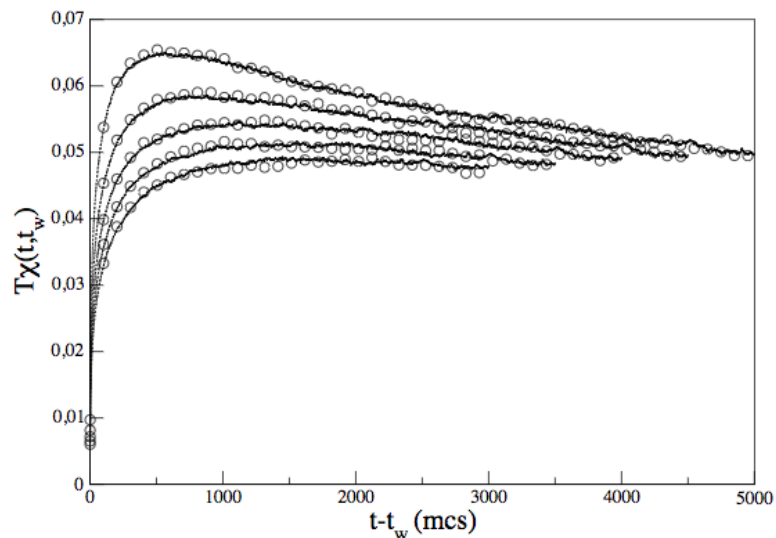
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Sub-critical coarsening in the MC dynamics of 2dIM

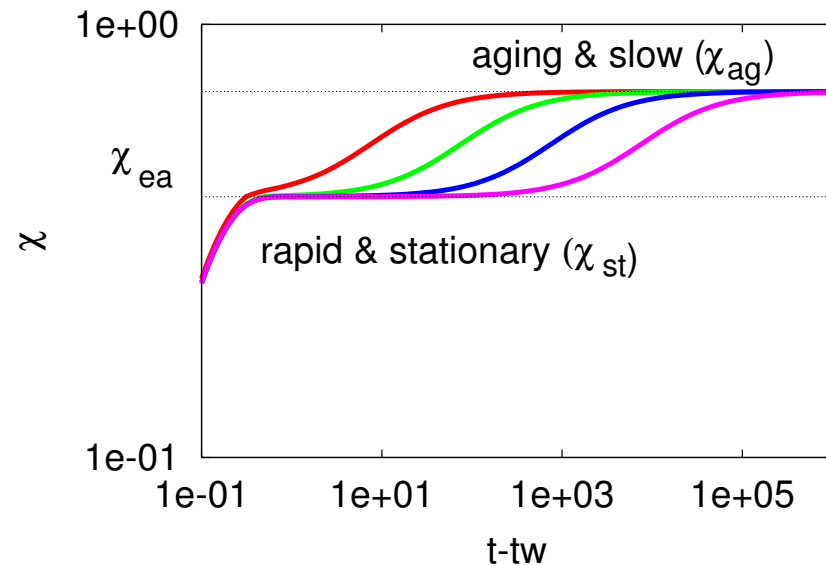


# Linear response

## Coarsening vs glassy



Lippiello, Corberi & Zannetti 05

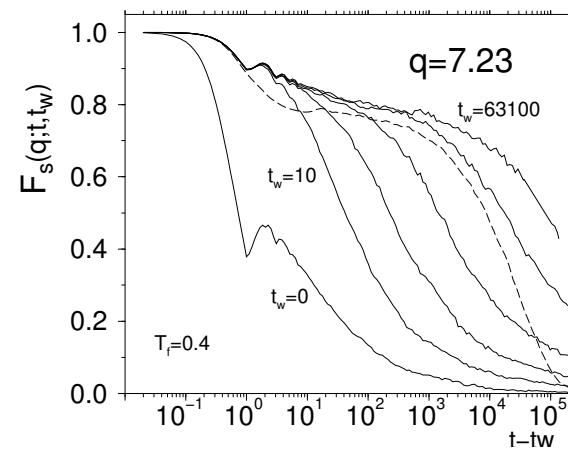
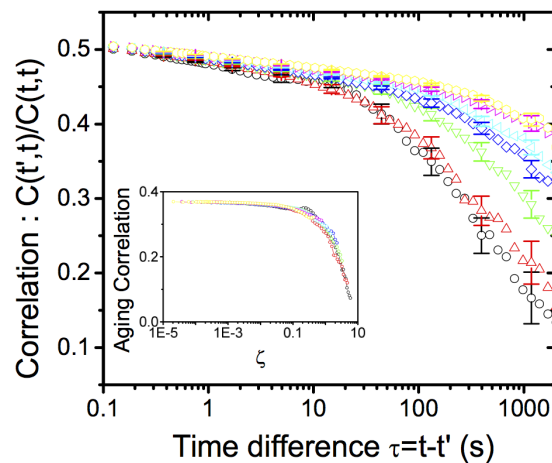
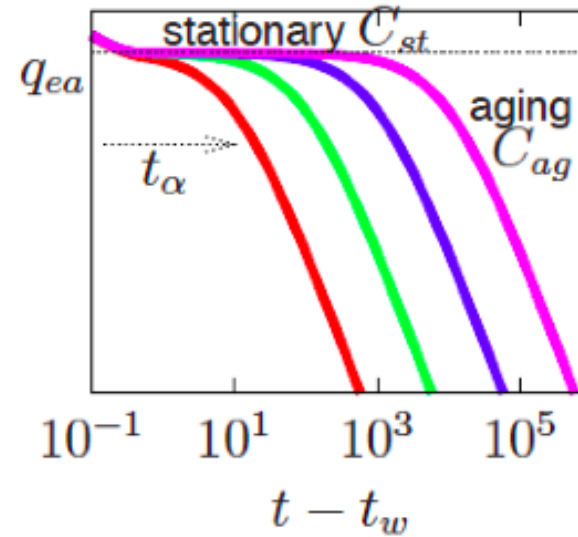
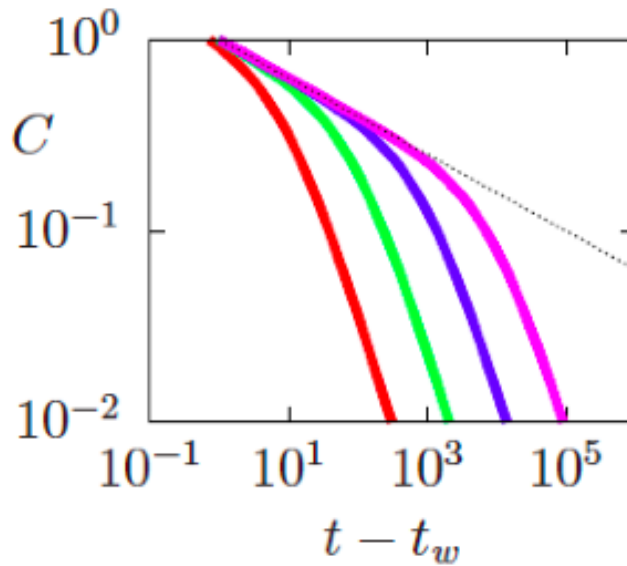


Sketch Chamon & LFC 07

There is no (weak) long-term memory in the coarsening problem. Just the stationary part will remain asymptotically, contrary to the sketch on the right for glasses & spin-glasses.

# Two-time self-correlation

## Comparison



Thiospinel (spin-glass)

Lennard-Jones mixture (glass)

---

# Plan of the lecture

---

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# The spherical $p = 2$ model

$$H = - \sum_{ij} J_{ij} s_i s_j + z \left( \sum_i s_i^2 - N \right)$$

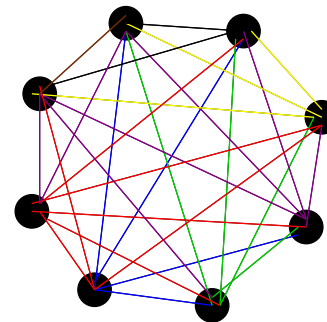
Fully connected interactions

Gaussian distributed

interaction strengths  $J_{ij}$

Spherical spins  $\sum_i s_i^2 = N$

$z$  is a Lagrange multiplier



$$\rho(\lambda_\mu) \propto \sqrt{(2J)^2 - \lambda_\mu^2}$$

$$H = - \sum_{\mu} \lambda_{\mu} s_{\mu}^2 + z \left( \sum_{\mu} s_{\mu}^2 - N \right)$$

Key: the largest eigenvalue

becomes **diffusive**,

$$\lambda_{\max} - z_{\infty} = 0$$

Same scaling laws for two-time corr. and resp. but no space dependence



---

# The $O(N)$ model

---

Upgrade the field to a vector  $\phi \mapsto \vec{\phi}$  with  $a = 1, \dots, N$  components

$$\vec{\phi} = (\phi_1, \dots, \phi_N)$$

The (over-damped) Ginzburg-Landau equation is now

$$\gamma_0 \partial_t \phi_a(\vec{r}, t) = - \frac{\delta \mathcal{F}[\vec{\phi}]}{\delta \phi_a(\vec{r}, t)} + \xi_a(\vec{r}, t)$$

The  $N \rightarrow \infty$  limit allows one to decouple the vector components :

$$\phi_a(\vec{r}, t) [\mu - \frac{1}{N} \sum_{b=1}^N \phi_b^2(\vec{r}, t)] \mapsto \phi_a(\vec{r}, t) z(t)$$

and the equations are now linear with a global constraint.

Coarsening is linked to the growth of the **diffusive**  $\vec{k} = 0$  mode.

---

# The $O(N)$ model

---

Upgrade the field to a vector  $\phi \mapsto \vec{\phi}$  with  $a = 1, \dots, N$  components

$$\vec{\phi} = (\phi_1, \dots, \phi_N)$$

The equations are now linear with a global constraint

$$\gamma_0 \partial_t \phi_a(\vec{r}, t) = \nabla^2 \phi_a(\vec{r}, t) + z(t) \phi_a(\vec{r}, t) + \xi_a(\vec{r}, t)$$

and

$$z(t) = \mu - N^{-1} \sum_a \phi_a^2(\vec{r}, t)$$

Solve for  $\phi_a(\vec{r}, t)$  as a function of  $z(t)$  and then impose the constraint to fix  $z(t)$ .

Coarsening is linked to the growth of the  $\vec{k} = 0$  mode, i.e. tendency to homogeneous order.

---

# Summary

---

- **At and below  $T_c$**  growth of equilibrium structures.

- The linear size of the equilibrium patches is measured by  $\mathcal{R}(t, T)$

- At  $T_c$  vanishing order parameter

Multiplicative scaling

$$C \simeq C_{eq} C_{ag}; \chi \simeq \chi_{eq} \chi_{ag}$$

- Below  $T_c$  non-vanishing order parameter

Additive scaling

$$C \simeq C_{eq} + C_{ag}; \chi \simeq \chi_{eq} + \chi_{ag}$$

- In both cases  $C_{ag}$  is finite while  $\chi_{ag}$  vanishes asymptotically.

**We shall discuss  $\chi$  and how it compares to  $C$  later.**

---

# Phase ordering kinetics

---

The lecture was about

- Growth of equilibrium patches at  $T_c$  and below  $T_c$ .
- Divergence of  $t_{eq}(L)$  with the system size.
- Existence of a single growing length  $\mathcal{R}(t, T)$
- Separation of time-scales and dynamic scaling, e.g.  $C = C_{eq} + C_{ag}$ .
- Two kinds of correlations : Space-time and two-time ones.
- **Dynamic universality classes** at and below  $T_c$ .
- The more tricky/rich **linear susceptibility**.

Is there a static growing length in all systems with slow dynamics ?

Which one ?

---

# Plan of the lecture

---

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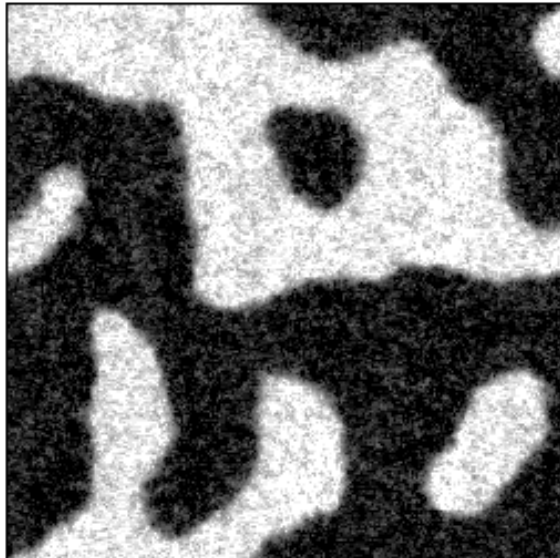
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# Multiplicative noise

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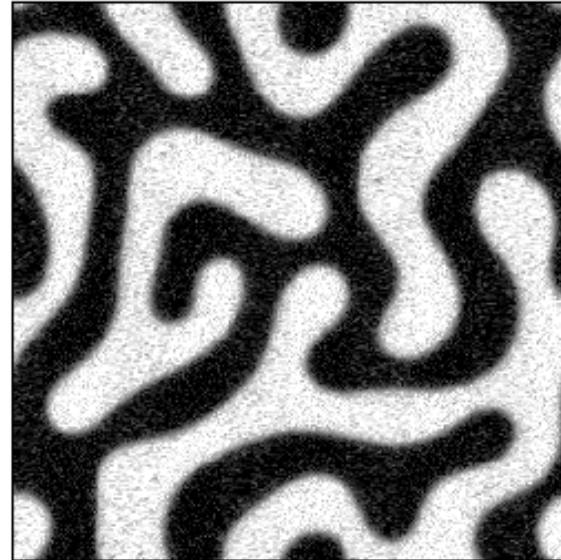
Numerical integration of the scalar field equations

NCOP



$$\mathcal{R}(t, T) \simeq t^{1/2}$$

COP



$$\mathcal{R}(t, T) \simeq t^{1/3}$$

---

# Voter model

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# Phenomenon

---

Similar questions can be asked in very well-known problems in math, *e.g.*

## Dynamics of a voter model starting from a random initial condition

- Purely dynamic, violation of detailed balance, no phase transition
- Two absorbing states
- The **dynamic mechanism** towards absorption is understood  
domain growth is driven by interfacial noise



---

# $2d$ Voter Model (VM)

---

Archetypical example of opinion dynamics

$H$  does not exist - kinetic model

$s_i = \pm 1$  Ising spins that

sit on the vertices of a lattice.

Voter update rule

choose a spin at random, say  $s_i$

choose one of its  $2d$  neighbours at random, say  $s_j$

set  $s_i = s_j$

In two dimensions full consensus, *i.e.*  $m = L^{-d} \sum_{i=1}^{L^d} s_i = \pm 1$  is reached in a timescale  $t_C \simeq L^2$  (with  $\ln L$  corrections)

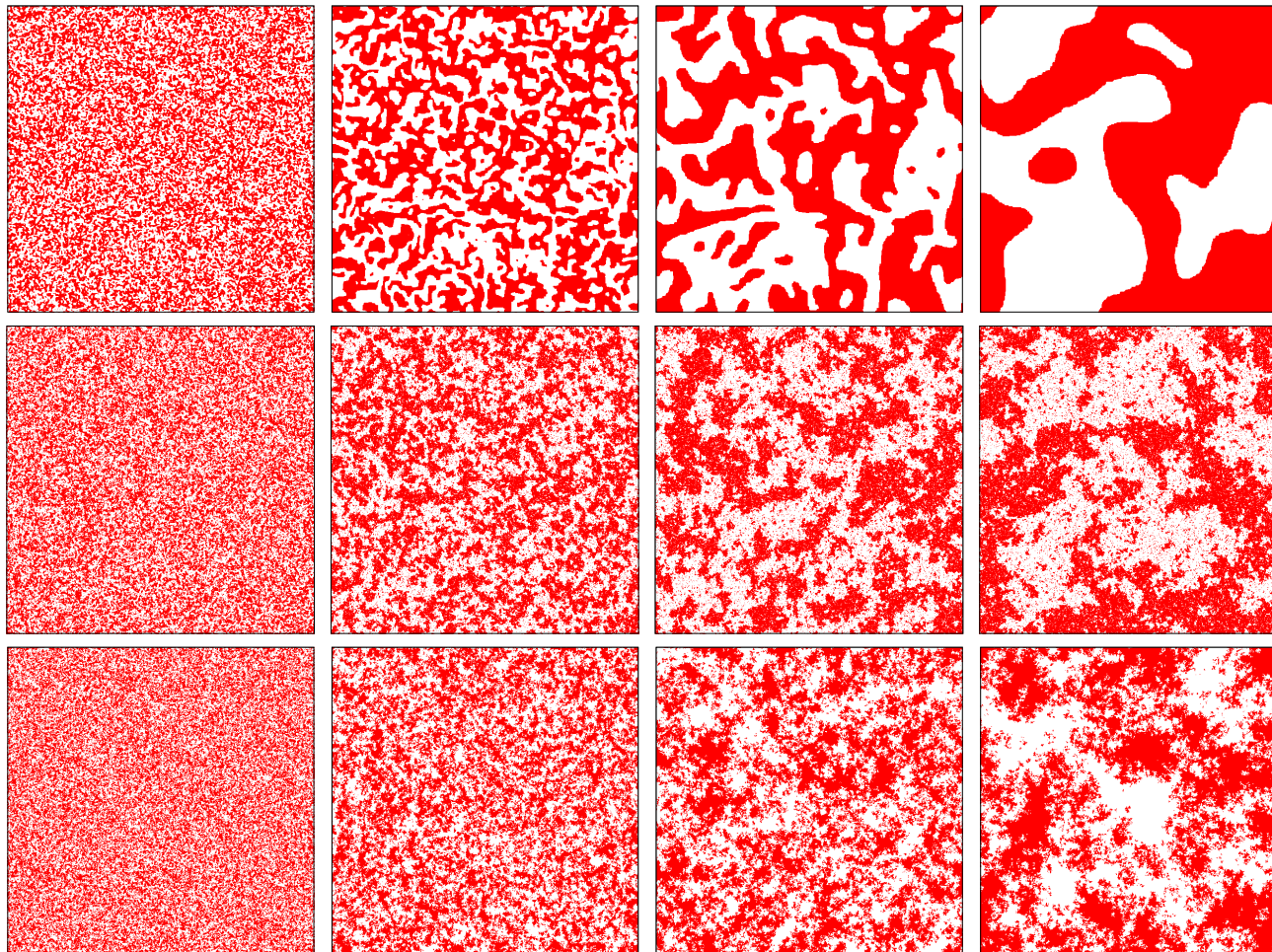
Clifford & Sudbury 73, Holley & Liggett 75, Cox & Griffeaths 86

---

# Phase ordering kinetics

---

$s_i = \pm 1$  at  $t = 0$  MCs, snapshots at  $t = 4, 64, 512, 4096$  MCs



Ising

$T = 0$

$T_c$

Voter

---

# Active matter dynamics

---

*2d* dumbbell system

$|\psi_{6i}|$  at  $\phi = 0.74$  and  $Pe = 10$  (coarsening towards co-existence)

---

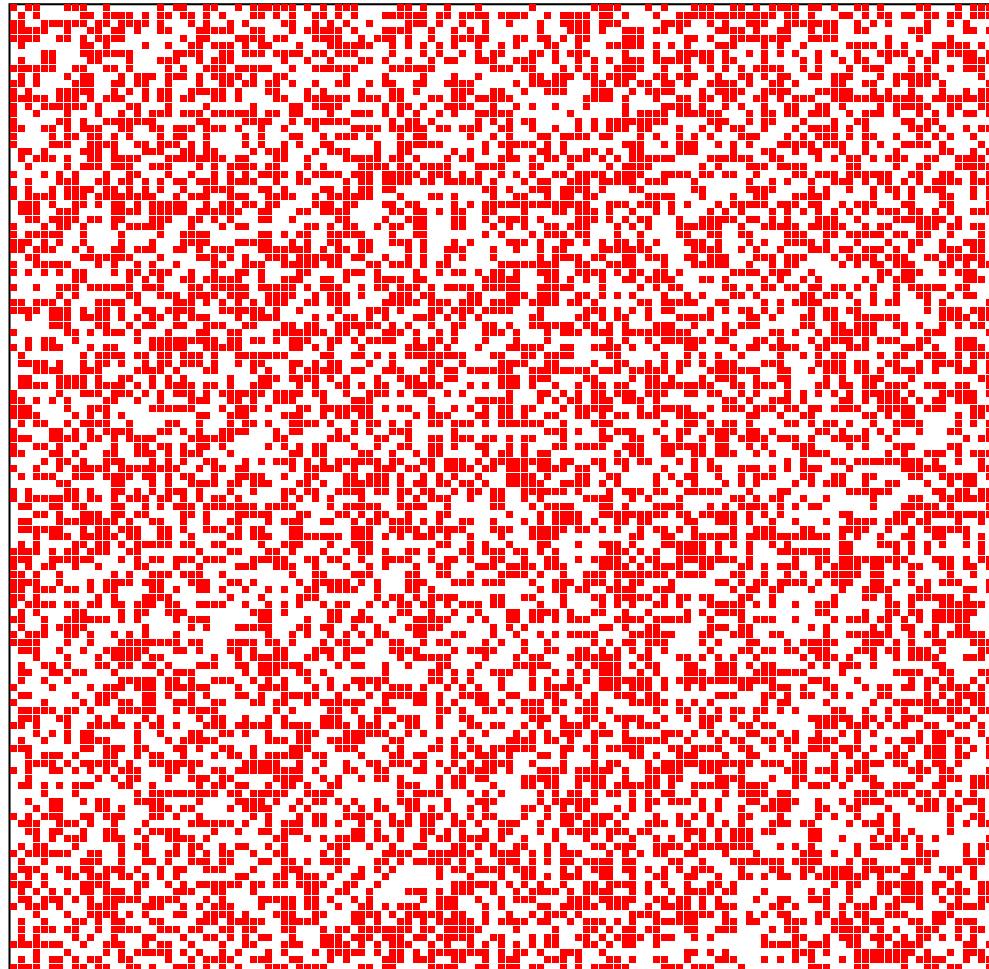
# Percolation issues

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---

# 2d square IM at $T=0$

---

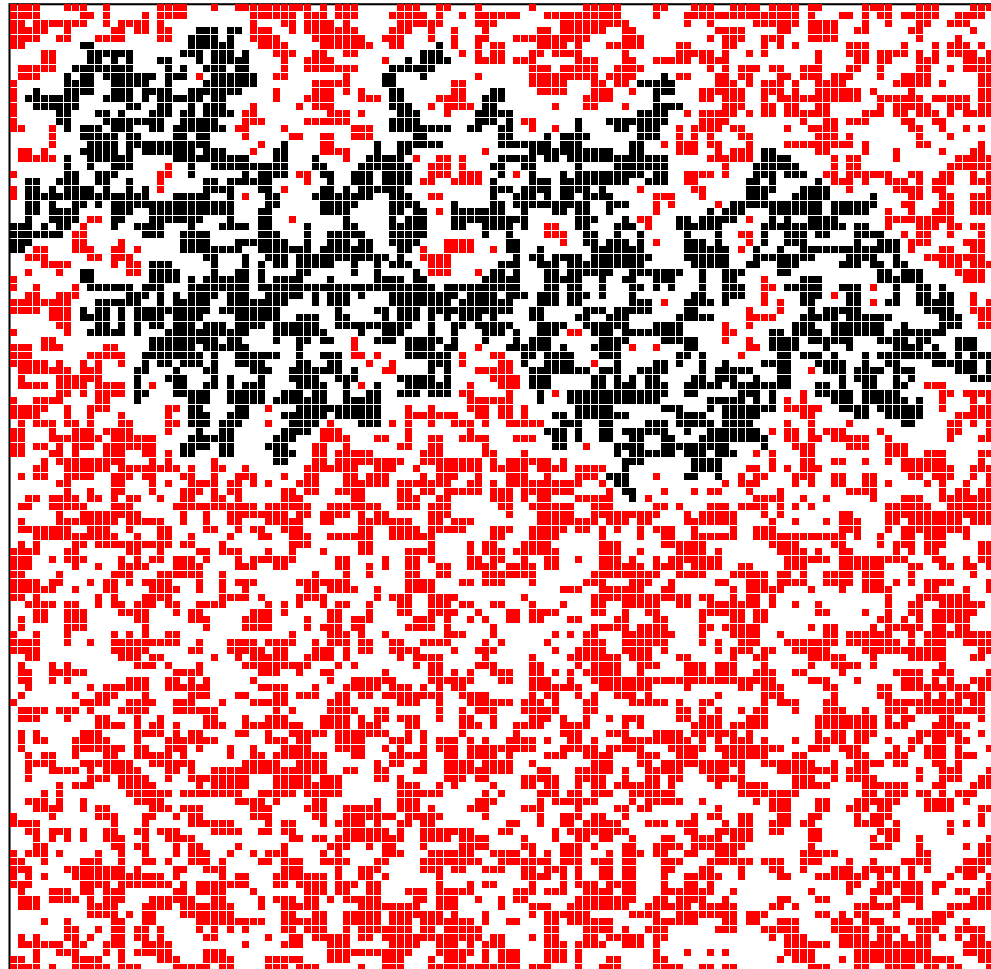


$t=0.0$

---

# 2d square IM at T=0

---

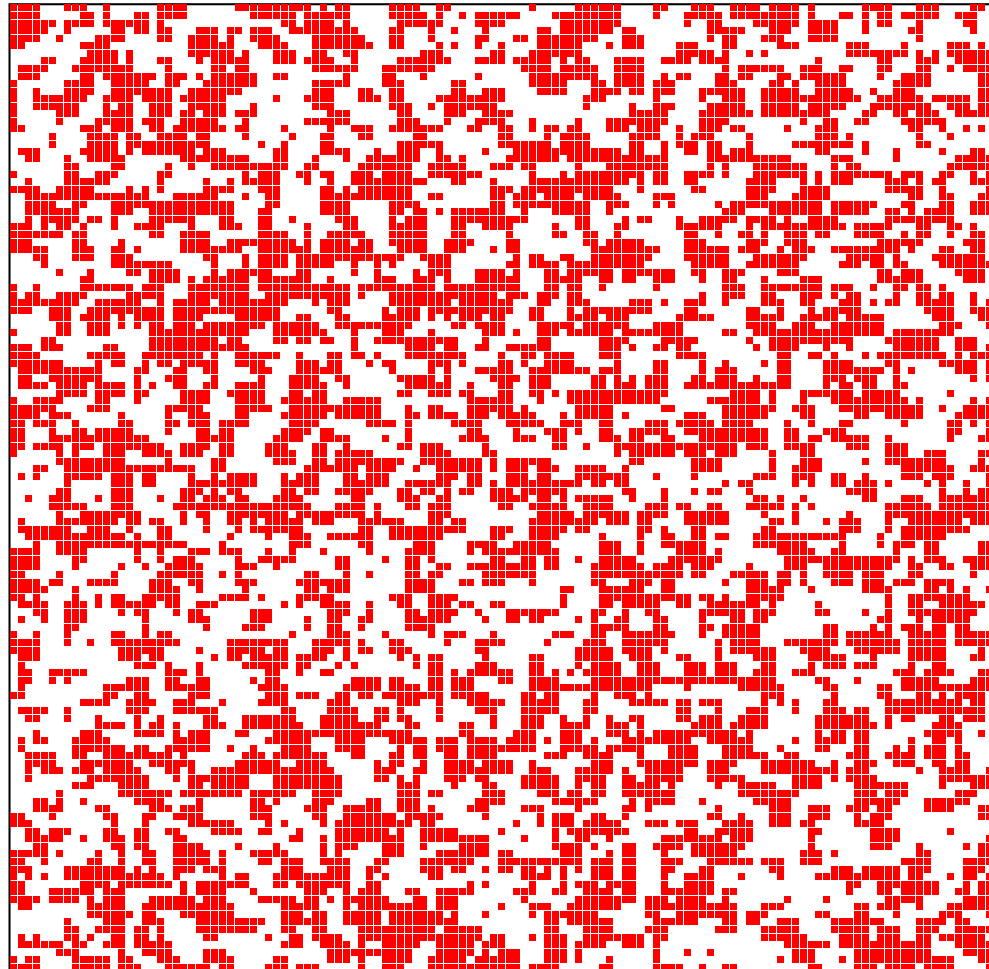


$t=0.57533$

---

# 2d square IM at T=0

---



$t=0.94844$

---

# 2d square IM at T=0

---



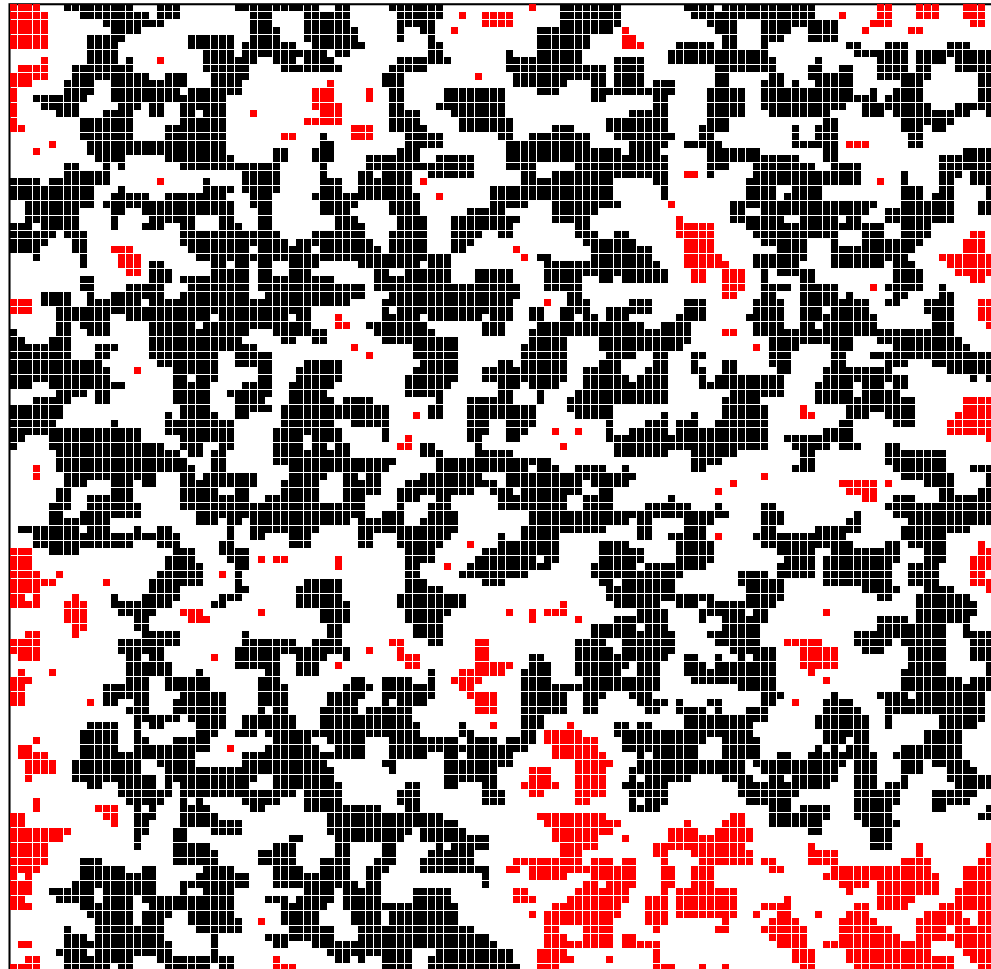
$t=2.00847$



---

# 2d square IM at T=0

---

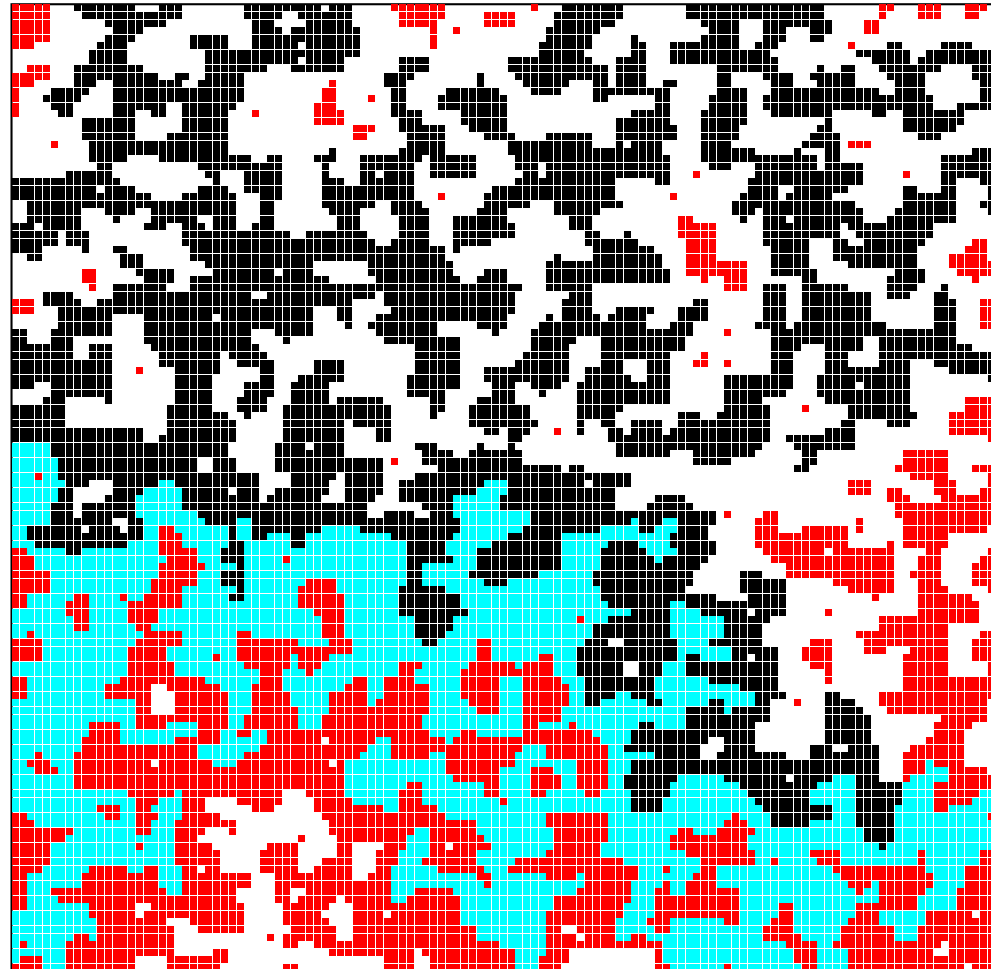


$t=2.57898$

---

# 2d square IM at T=0

---

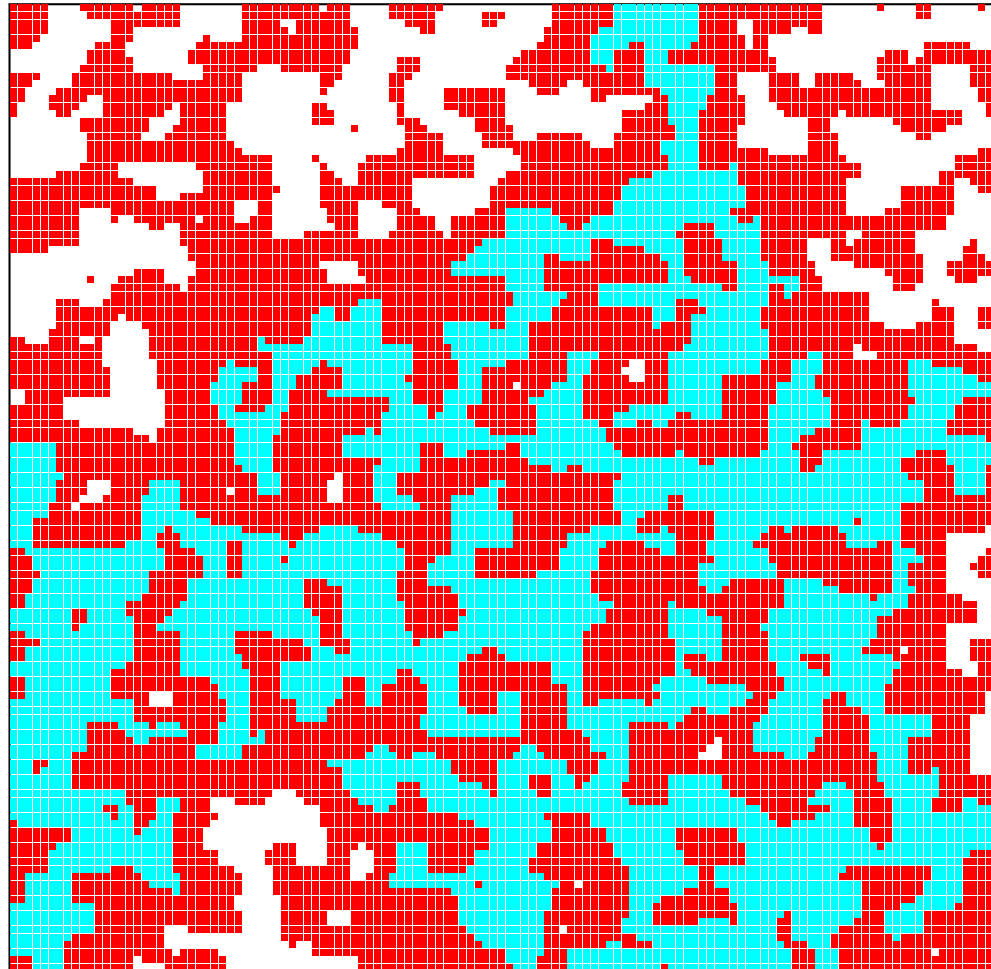


$t=3.99211$

---

# 2d square IM at T=0

---

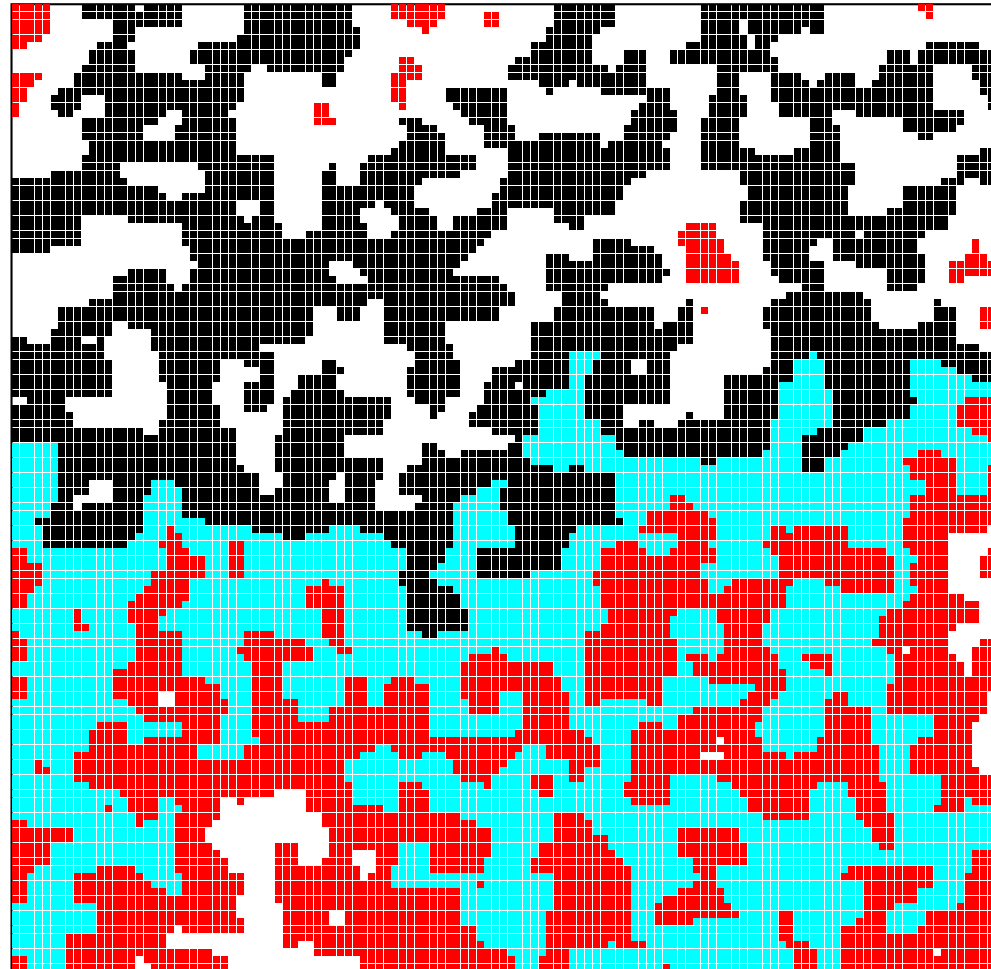


$t=6.58423$

---

# 2d square IM at T=0

---



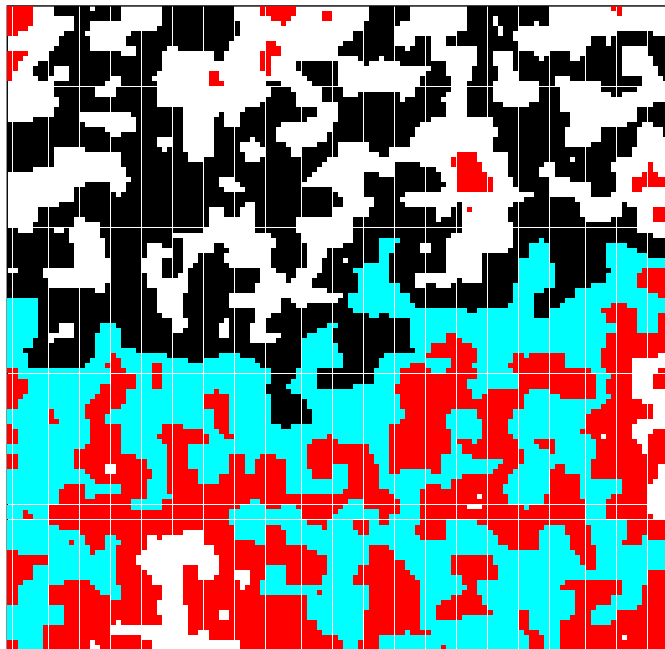
$t=7.46144$

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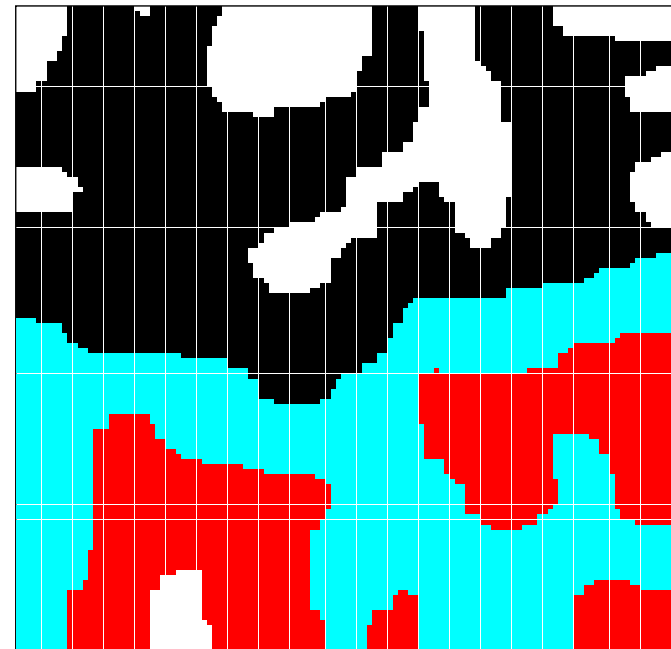
# 2d square IM at $T=0$

---

The percolating structure was decided at  $t_p \simeq 8$  MCs



$t=7.46144$



$t=128.0$

---

# Complex field & cold atoms

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---

# Complex field theory in $3d$

---

Relativistic bosons;  $^4\text{He}$ , type II superconductors, cosmology, etc.

$$-c^{-2}\ddot{\psi} + \nabla^2\psi + 2i\mu\dot{\psi} = g(\psi^2 - \rho)\psi$$

$c$  is the velocity of light,  $\rho$  and  $g$  parameters in (Mexican hat) potential.

Limits

$\mu \rightarrow 0$  :

$$-c^{-2}\ddot{\psi} + \nabla^2\psi = g(|\psi|^2 - \rho)\psi$$

Goldstone

$c \rightarrow \infty$  :

$$2i\mu\dot{\psi} + \nabla^2\psi = g(|\psi|^2 - \rho)\psi$$

Gross-Pitaevskii

models

---

# Complex field theory in 3d

---

Relativistic bosons;  ${}^4\text{He}$ , type II superconductors, cosmology, etc.

$$-c^{-2}\ddot{\psi} + \nabla^2\psi + 2i\mu\dot{\psi} = g(\psi^2 - \rho)\psi$$

The energy functional

$$E = \int d^3x \left( c^{-2}|\dot{\psi}|^2 + |\vec{\nabla}\psi|^2 - g\rho\psi^2 + g\psi^4 \right)$$

is conserved under the dynamics.

The energy is minimised by the static configuration  $\psi = \sqrt{\rho} e^{i\chi}$  with  $\chi = ct$

There are static vortex solutions, e.g.  $\psi(\vec{x}) = f(r) e^{in\theta}$  with  $f(0) = 0$  and  $f(r \rightarrow \infty) = \sqrt{\rho}$ , and  $n \in \mathbb{Z}$  (thin tubes at the centre of which the field vanishes and the phase turns around).

**Tsubota, Kasamatsu & Kobayashi 13, Kobayashi & Nitta 15, etc.**



# Complex field theory in 3d

Stochastic noise and dissipation added

$$-c^{-2}\ddot{\psi} + \nabla^2\psi + 2i\mu\dot{\psi} - \gamma\dot{\psi} = g(\psi^2 - \rho)\psi - \sqrt{\gamma T}\xi$$

Langevin-like dynamics

$-\gamma$  viscosity,  $\xi$  complex Gaussian white noise in normal form

$$\langle \xi_i(\vec{x}, t) \rangle = 0 \text{ and } \langle \xi_i(\vec{x}, t_1) \xi_j(\vec{y}, t_2) \rangle = \delta_{ij} \delta^{(3)}(\vec{x} - \vec{y}) \delta(t_1 - t_2)$$

Passage to Fokker-Planck formalism allows to show that the dynamics takes the system to

$$\lim_{t \rightarrow \infty} P(\psi, t) = P_{GB}(\psi) \propto e^{-\beta E}$$

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# Complex field theory in 3d

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Relativistic bosons;  $^4\text{He}$ , type II superconductors, cosmology, etc.

$$-c^{-2}\ddot{\psi} + \nabla^2\psi + 2i\mu\dot{\psi} - \gamma\dot{\psi} = g(\psi^2 - \rho)\psi - \sqrt{\gamma T}\xi$$

Langevin-like dynamics

$-\gamma$  viscosity,  $\xi$  Gaussian white noise in normal form

In the limit  $c \rightarrow \infty$ , the stochastic Gross-Pitaevskii equation

$$(2i\mu - \gamma)\dot{\psi} = -\nabla^2\psi + g(\psi^2 - \rho)\psi + \sqrt{\gamma T}\xi$$

Gardiner et al 00s

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# 3d XY lattice model

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Archetypical classical magnetic example

$$H = -J \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j$$

$J > 0$  ferromagnetic coupling constant.

$\langle ij \rangle$  sum over nearest-neighbours on a 3d lattice

$\vec{s}_i$  planar spins: two components with constant modulus  $\Rightarrow$  angle  $\theta_i$ .

Second order phase transition with spontaneous symm breaking at  $T_c > 0$ .

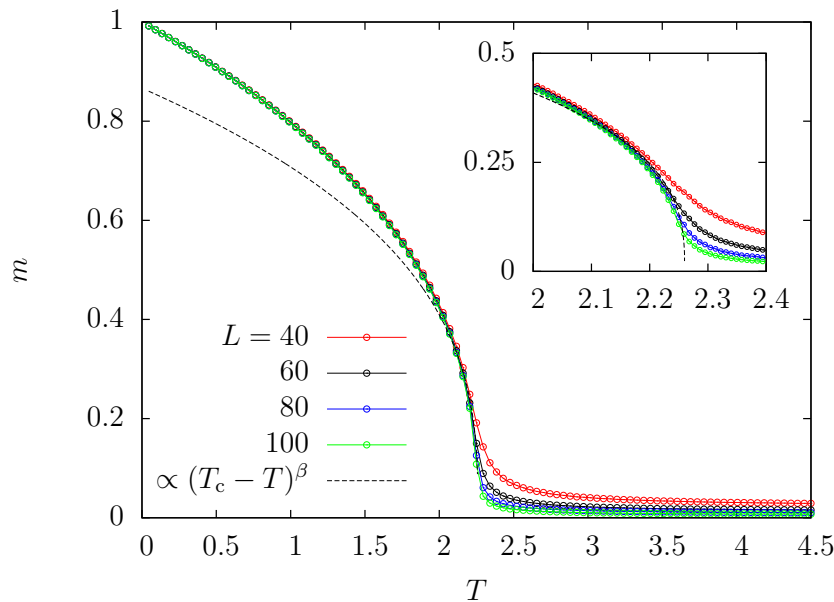
Order parameter: spin-alignment,  $\vec{m} \equiv N^{-1} \sum_i \langle \vec{s}_i \rangle$ .

No intrinsic spin dynamics, **Monte Carlo rules** mimic coupling to thermal bath.

**Non-conserved order parameter dynamics** [  $\uparrow\downarrow$  towards  $\uparrow\uparrow$  ] etc. allowed.

# Statics

## Phase transition and order parameter in the field equation



$$L^3 m = \left| \sum_{ijk} \langle \psi_{ijk} \rangle \right|$$

critical temperature

$$T_c = 2.26$$

critical exponent

$$\beta = 0.347$$

**Kobayashi & LFC 16**

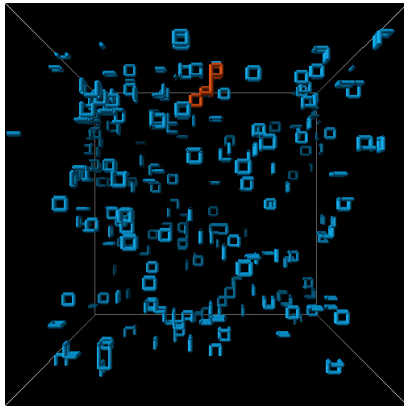
$T_c$  and critical exponents from kurtosis (Binder parameter), susceptibility, specific heat, etc. Values compatible w/results from simulations **Ballesteros et al. 96**, **Hasenbusch & Török 99** and  $\epsilon$  expansion **Guida & Zinn-Justin 98**, **Täuber & Diehl 14** for models in the same universality class.

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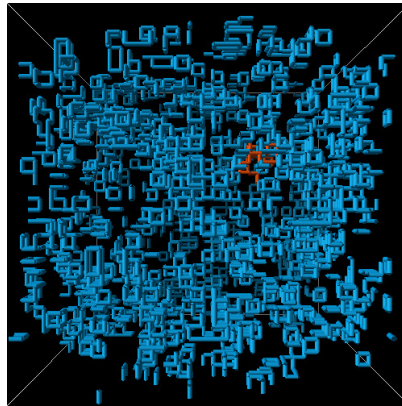
# Vortex configurations

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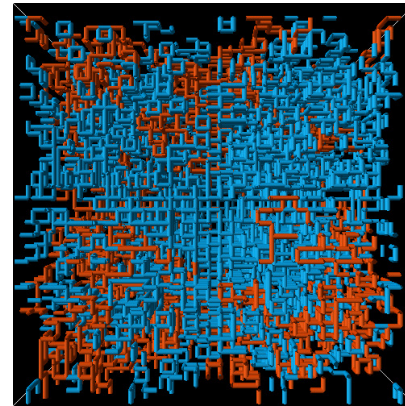
In equilibrium



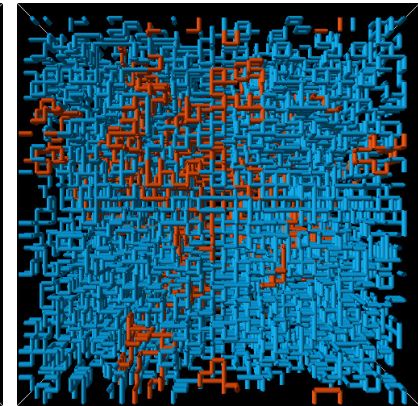
$0.6 T_c$



$0.8 T_c$



$T_c$



$1.2 T_c$

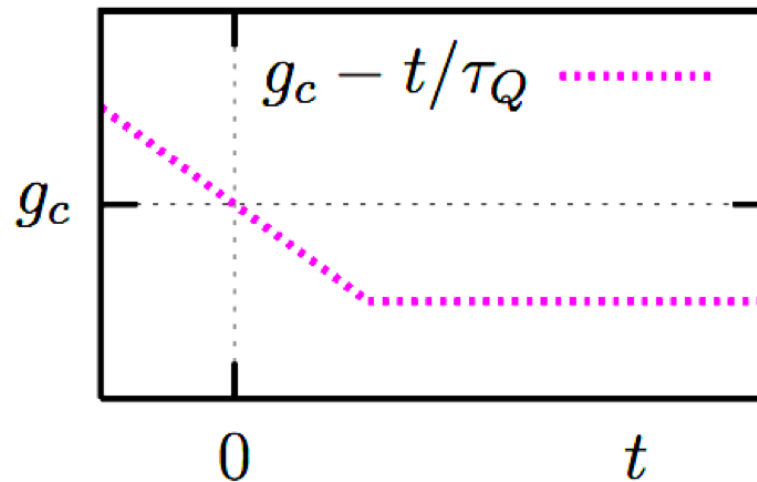
Periodic boundary conditions (torus) implies that the vortex lines are closed, i.e. loops.

Stochastic reconnection rule.

All vortex loops in blue, the longest one in red.

# Dynamics after a quench

with  $g$  the control parameter



In the picture: annealing with finite rate.

Infinately fast quench:  $T \gg T_c$  for  $t < 0$  and  $T = 0$  for  $t > 0$

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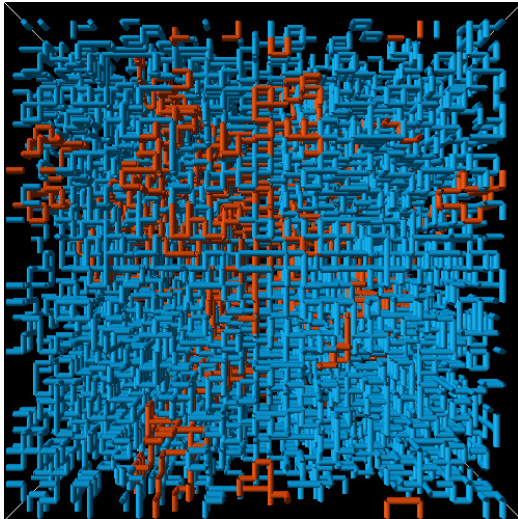
# Complex field theory in $3d$

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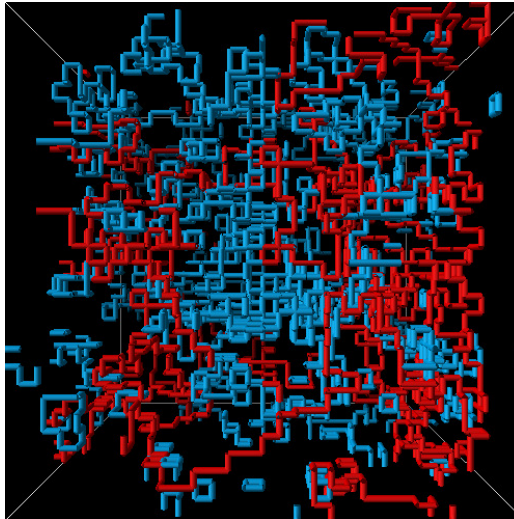
Progressive elimination of vortex loops after a quench

$T \gg T_c$

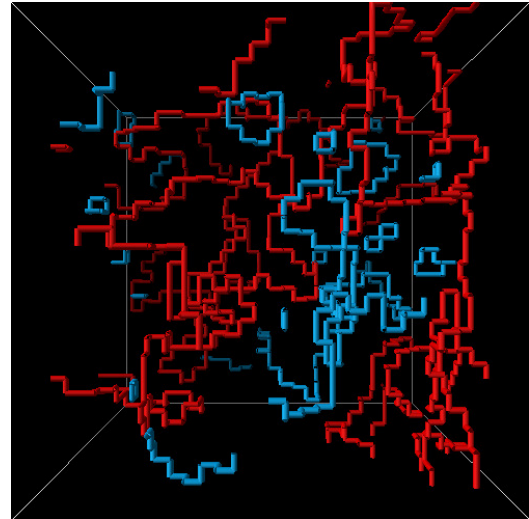
$T = 0$



$t = 0$



$t = 3$



$t = 5$

As  $\rho_{\text{vortex}} \downarrow$  the reconnection rule loses importance

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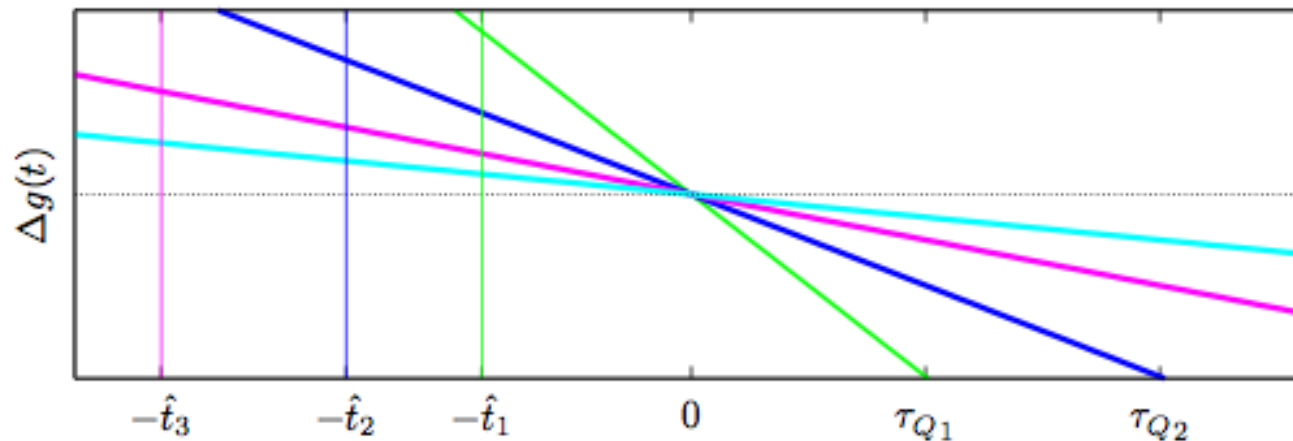
# **Slow cooling & Kibble-Zurek**

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# Finite rate quenching protocol

How is the scaling modified for a very slow quenching rate?



$$\Delta g \equiv g(t) - g_c = -t/\tau_Q \quad \text{with} \quad \tau_{Q1} < \tau_{Q2} < \tau_{Q3} < \tau_{Q4}$$

Standard time parametrization

$$g(t) = g_c - t/\tau_Q$$

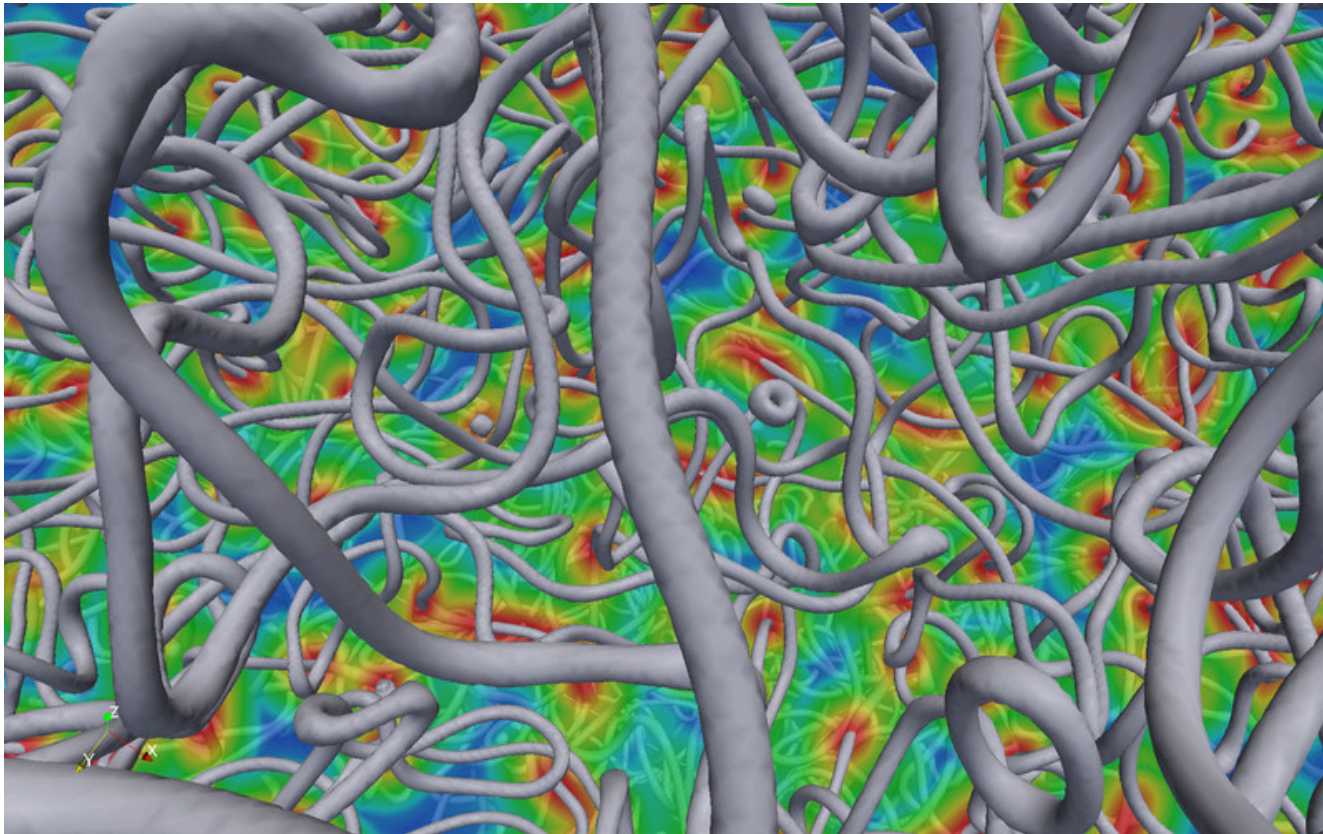
Simplicity argument: linear cooling could be thought of as an approximation of any cooling procedure close to  $g_c$ .

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# Theoretical motivation

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## Network of cosmic strings



They should affect the Cosmic Microwave Background, double quasars, etc.

**Picture from M. Kunz's group (Université de Genève)**

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# Topological defects

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instantaneous configurations



Domain walls in the  $2d$ IM



Vortices in the  $3d$  xy model

One can give a precise mathematical definition but the visual one is enough

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# Density of topological defects

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## Kibble-Zurek mechanics for 2nd order phase transitions

### The three basic assumptions

- Defects are **created** close to the critical point.
- Their density in the ordered phase is inherited from the value it takes when the system falls out of equilibrium on the **symmetric** side of the critical point. It is determined by

Critical scaling above  $T_c$

- The **dynamics in the ordered phase** is so slow that it can be **neglected**.

### and one claim

- results are **universal**.

**that we critically revisited within 'thermal' phase transitions**



# Topological defects

after an instantaneous quench : dynamic scaling



$$\Delta n(t) \simeq [\mathcal{R}(t, T)]^{-d} \simeq [\lambda(T(t))]^{-d} t^{-d/z_d}$$

Remember the initial ( $g \rightarrow \infty$ ) configuration: already there !