EQUILIBRATION, INTEGRABILITY & GENERALIZED GIBBS ENSEMBLES

(IN DISORDERED SYSTEMS)

- DISORDER OF THE USUAL KIND

  $J_{i_1 \cdots i_p}$ FROM $P(J_{i_1 \cdots i_p})$

  PARAMETERS DRAWN FROM A pdf

QUANTUM QUENCHES & EQUILIBRATION

SETTING

1) INITIALIZE YOUR QUANTUM SYSTEM

- IN A STATE $|\psi_0\rangle$ OR
- IN A MIXTURE $\hat{\rho}_0$

WHICH KNOW ABOUT A HAMILT $\hat{H}_0$

2) EVOLVE THIS STATE IN ISOLATION WITH A DIFFERENT HAMILT $\hat{H}$

$$\hat{U} = e^{-i\hat{H}t/\hbar}$$

3) ASK : WHAT HAPPENS IN THE LIMIT

$$\lim_{t\to\infty} \quad \lim_{N\to\infty} \quad ?$$

N: # d.o.f.

SPECIALLY w/ LOCAL OPERATORS

$|\psi_0\rangle \longrightarrow |\psi\rangle$

$|\psi\rangle$ NOT NECESS AN EIGENST OF $\hat{H}$

TAKE THE CASE IN WHICH THE INITIAL STATE IS JUST $|\psi_0\rangle$ EIGENST OF $\hat{H}_0$

TYPICALLY ONE CHOOSES $|\psi_0\rangle = |\psi_{GS}^{\hat{H}_0}\rangle$

$$E_0 = \langle \psi_0 | \hat{H}_0 | \psi_0 \rangle \qquad \text{AT } t = 0^-$$

$$E = \langle \psi_0 | \hat{H} | \psi_0 \rangle \qquad \text{AT } t = 0^+$$

AND CONSERVED EVEN AFTER

WITH THE QUENCH ENERGY INJECTION (OR EXTRACTION)

WHERE IS IT IN THE SPECTRUM OF $\hat{H}$?

FOR NON-INTEGRABLE SYSTEMS ETH

DEUTSCH, SREDNICKI 90s

FOR INTEGRABLE ONES GGE

CALABRESE & CARDY, OLSHANII, DUNJKO, RIGOL .

WHAT ARE INTEGRABLE SYSTEMS ?

NAIVELY PBM FOR WHICH WE CAN WRITE AN EXACT SOL.

$\#$ d.o.f. $\Rightarrow$ N CHARGES $\hat{I}_\mu$

$$[\hat{H}, \hat{I}_\mu] = 0 \qquad \mu = 1, \dots, N$$

$$[\hat{I}_\mu, \hat{I}_\nu] = 0 \qquad \mu \neq \nu$$

"ORTHOGONALITY"

CLAIM GGE $\lim_{N\to\infty}$ FIRST

$$|\psi\rangle = \lim_{t\to\infty} |\psi(t)\rangle$$

(SCHRÖD. PICTURE)

$$A_\infty = \langle\psi| \hat{A} |\psi\rangle$$

FOR A "NON-PATHOL." $\hat{A}$
AND "LOCAL" (IN SOME SPACE)

THIS IS DETERMINED BY THE UNITARY EVOLUTION AND

SHOULD COINCIDE WITH

$$\langle \hat{A} \rangle_{GGE} = \mathrm{Tr}\left(\hat{A}\, \hat{\rho}_{GGE}\right)$$

WHERE

$$\hat{\rho}_{GGE} = \frac{1}{Z_{GGE}}\, e^{-\sum_{\mu=1}^{N} \gamma_\mu \hat{I}_\mu}$$

AND γ_μ FIXED BY

$$\langle \psi_0 | \hat{I}_\mu | \psi_0 \rangle = \mathrm{Tr}\left(\hat{I}_\mu \hat{\rho}_{GGE}\right)$$

$$\mathcal{I}_\mu = \langle \hat{I}_\mu \rangle_{GGE}$$

INITIAL VALUES

AVERAGE OVER $\hat{\rho}_{GGE}$

"NATURAL" EXTENSION OF THE GIBBS-BOLTZMANN FORM, WHICH INCLUDES ALL CHARGES (N OF THEM) AND NOT ONLY $\hat{H}$

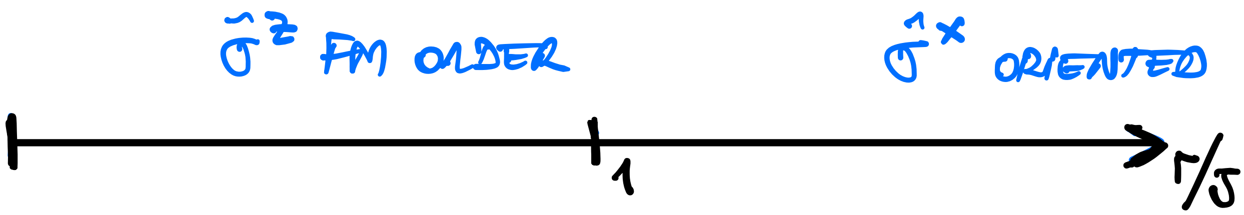
NB: MICRO CANONICAL

A QUANTUM INTEGRABLE MODEL

$$\hat{H} = -J \sum_i \hat{\sigma}_i^z \sigma_{i+1}^z + \Gamma \sum_i \hat{\sigma}_i^x$$

SU(2) $\quad [\hat{\sigma}_i^a, \hat{\sigma}_i^b] = 2i\,\varepsilon^{abc}\,\hat{\sigma}_i^c$

PAULI MATRICES



WHAT ARE THE $\hat{I}_\mu$?

JORDAN-WIGNER (SPINS → FERMIONS) + BOGOLIUBOV
($i \to k$)

$$\hat{H} = \sum_k \epsilon_k(\Gamma/J)\, \hat{c}_k^\dagger \hat{c}_k$$

$$= \sum_k \epsilon_k(\Gamma/J)\, \hat{n}_k$$

→ FERMION #

$$\text{GGE with } \hat{n}_k \quad \Leftarrow \quad \begin{cases} [\hat{H}, \hat{n}_{k'}] = 0 & \forall k' \quad \text{FERMION \#} \\ [\hat{n}_k, \hat{n}_{k'}] = 0 & \forall k, k' \end{cases}$$

CONSTRUCT GGE & CHECK IT WORKS ✓

THIS IS A PARTICULARLY SIMPLE CASE SINCE NON-INTERACTING ⇒ DIAGONALIZ.

INTERACTING ONES MORE DIFFICULT

HOW MUCH OF THIS IS DUE TO QUANTUM CHARACTER AND HOW MUCH TO INTEGRABILITY ?

$\underbrace{\text{SIMILAR SETTING}}$ IN CLASSICAL MODELS

"QUANTUM QUENCHES"

- START FROM $\psi_0 = \{ x_i(0), p_i(0) \}_0^{i=1,\dots,N}$ DRAWN FROM

  $P(\psi_0)$ WHICH KNOWS ABOUT H_0

  eg. GROUND STATE OR

  $$P(\psi_0) \propto e^{-\beta_0 H_0(\psi_0)}$$

- EVOLVE WITH $H \neq H_0$ AND NEWTON DYNAMICS

  $\{ x_i(t), p_i(t) \}^{i=1,\dots,N}$ TRAJECTORIES IN PHASE SPACE

USUALLY N # PART IS CONST

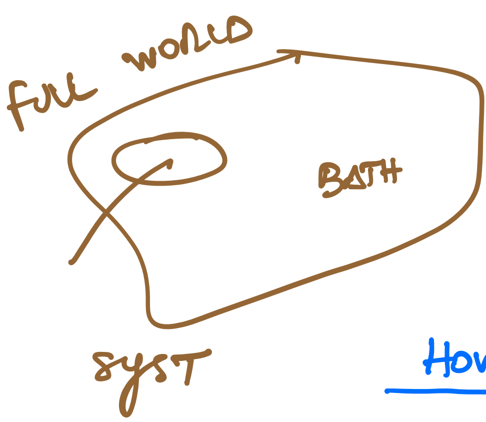
$E = H(\psi_0)$ IS CONST TOO

HOW DO WE GO FROM MICRO TO CANONICAL?

USUALLY MICROCANONICAL MEASURE
ASSUMED FOR CASES WITH E CONST
(MAYBE ALSO A FEW OTHER CONST OF MOTION)

$$\rho_{\text{MICRO}}(\mathcal{C}) \propto \delta(H(\mathcal{C}) - E)$$

FLAT DISTRIB. $\Longrightarrow$ ALL CONF EQUALLY PROBABLE



$$\rho_{\text{CAN}}^{\text{SYST}} \propto e^{-\beta H_{\text{SYST}}}$$

HOW TO DERIVE IT ?

$$E = E_{\text{SYST}} + E_{\text{INT}} + E_{\text{BATH}}$$

ASSUMPTIONS $\quad E_{\text{INT}} \ll E_{\text{SYST}} \ll E_{\text{BATH}}$

$$\beta = \frac{\partial S_{\text{BATH}}}{\partial E_{\text{BATH}}} = \Theta(1) = \text{CONST.}$$

SHORT-RANGE INTERACTIONS - PBMS FOR LONG RANGE
CAMPA, DAUXOIS, RUFFO 10s. TATA STATPHYS SATELLITE

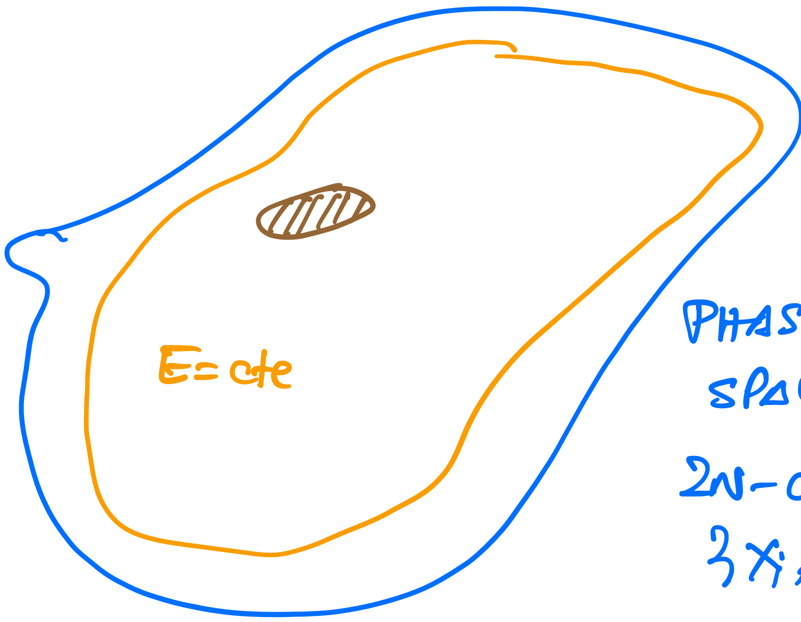
IF <u>INTEGRABLE MODEL</u>

$I_\mu \quad \mu = 1, \ldots, N$ CONST OF MOTION

$\{H, I_\mu\} = 0$

POISSON BRACKETS

$\{I_\mu, I_\nu\} = 0$



E = CONST.
2N-1 dim.

PHASE SPACE

2N-dim
$\{x_i, p_i\}$

 I_μ = CONST.

MUCH MORE CONSTRAINED
N-dim ONLY
STILL LARGE THOUGH.

d.o.f. $N = \#$ CONST MOTION

MEASURES IN CLASSICAL INTEGRABLE SYST

$$\rho^{\text{MICRO}}(\{I_\mu\}) = c \prod_{\nu=1}^{N} \delta\left(I_\nu(\{\vec{x},\vec{p}\}) - I_\nu\right)$$

$\downarrow$

VALUES OF CONST OF MOTION

YUZBASHYAN 16

CONDITIONS TO DERIVE CANONICAL FROM MICRO

FOLLOWING CAMPA, DAUXOIS, RUFFO

$$I_\mu \longrightarrow I_\mu^{\text{SYST}} + I_\mu^{\text{BATH}} \qquad \text{ADDITIVITY}$$

$$I_\mu = \Theta(N) \qquad \text{EXTENSIVITY}$$

TO BE ABLE TO, FOR EACH I_μ, DROP I_μ^{INT} AND ARGUE $I_\mu^{\text{SYST}} \ll I_\mu^{\text{BATH}}$

NOT OBVIOUS IF N I_μ's

STILL PROPOSE $\quad \rho_{\text{GGE}} \propto e^{-\sum_\mu \gamma_\mu I_\mu}$

WHY LOOK AT THE CLASSICAL CASE?

1 - SOME PECULIAR FEATURES ARE NOT ESSENTIALLY QUANTUM

2 - WE CAN STUDY AN INTERACTING INTEGRABLE MODEL IN MANY WAYS (ANALYTICALLY)

3 - WE CAN STUDY A "COMPLEX" DYN PHASE DIAGRAM

4 - NICE CONNECTION W/ DISORDERED SYSTEMS AND A MODEL WE KNOW VERY WELL

THE SPHERICAL ($p=2$) SK

D. BARBIER, LFC, LOZANO, NESSI, PICCO & TARTAGLIA ~ 18-21

ALSO SPOHN ET AL, GOLDFRIEND & KURCHAN

TODA LATTICE

WHICH IS THE QUESTION ?

$$\underbrace{\lim_{t\to\infty} \quad \lim_{N\to\infty}}$$

$$\overline{\langle A(t)\rangle}_{ic} = \lim_{\tau\to\infty} \frac{1}{\tau} \int_t^{t+\tau} dt' \langle A(t')\rangle_{ic}$$

$$\overset{?}{\parallel}$$

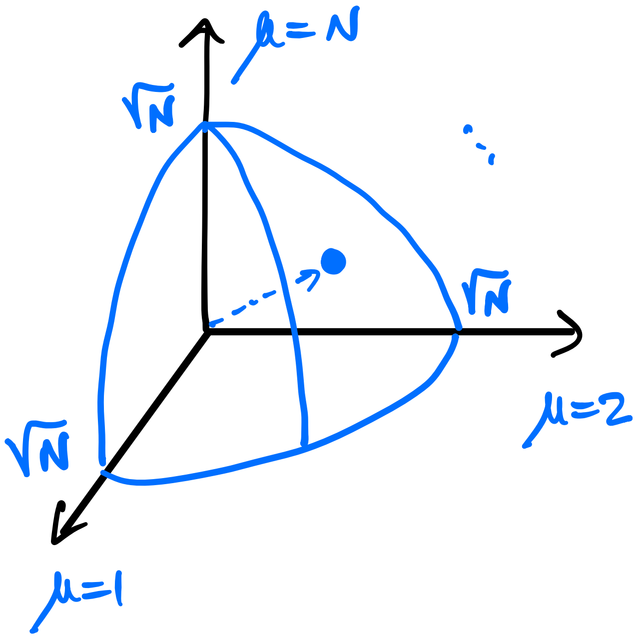
$$\langle A\rangle_{GGE} = \int d^N x\, d^N p \; A(\vec{x},\vec{p}) \; \rho_{GGE}(\vec{x},\vec{p})$$

WITH

$$\rho_{GGE}(\vec{x},\vec{p}) = \frac{1}{Z_{GGE}} e^{-\sum_\mu \gamma_\mu I_\mu(\vec{x},\vec{p})}$$

NB: GOING FROM A "MICRO" PERSP. TO A "CANONICAL" ONE

NEUMANN'S MODEL (1850)



A CLASS. PART CONSTRAINED TO MOVE ON A SPHERE

$$\sum_\mu x_\mu^2 = N$$

RADIUS $\sqrt{N}$

$$H = \underbrace{\sum_\mu \frac{p_\mu^2}{2m}}_{\text{KINETIC ENERGY}} \underbrace{- \frac{1}{2} \sum_\mu \lambda_\mu x_\mu^2}_{\text{HARMONIC POTENTIAL ENERGY}}$$

ADD $\quad - z(\vec{x}, \vec{p}) \sum_\mu x_\mu^2$

$z(\vec{x},\vec{p})$ IMPOSES THE SPH CONST

$$= 2 H_{kin}(\vec{x},\vec{p}) - 2 H_{pot}(\vec{x},\vec{p})$$

K. UHLENBECK 80s

ON EACH TRAJECTORY

$$I_\mu(\vec{x},\vec{p}) = x_\mu^2 + \frac{1}{mN} \sum_{\nu(\neq\mu)} \frac{(x_\mu p_\nu - x_\nu p_\mu)^2}{\lambda_\nu - \lambda_\mu}$$

ARE THE N- CONST OF MOTION SATISFYING

$$\{ I_\mu, I_\nu \} = 0$$

$$\sum_\mu I_\mu = N \qquad \sum_\mu \lambda_\mu I_\mu = -2H$$

INTEGRABLE SYST PEOPLE

D. BERNARD, AVAN, BABELON, TALON 90s

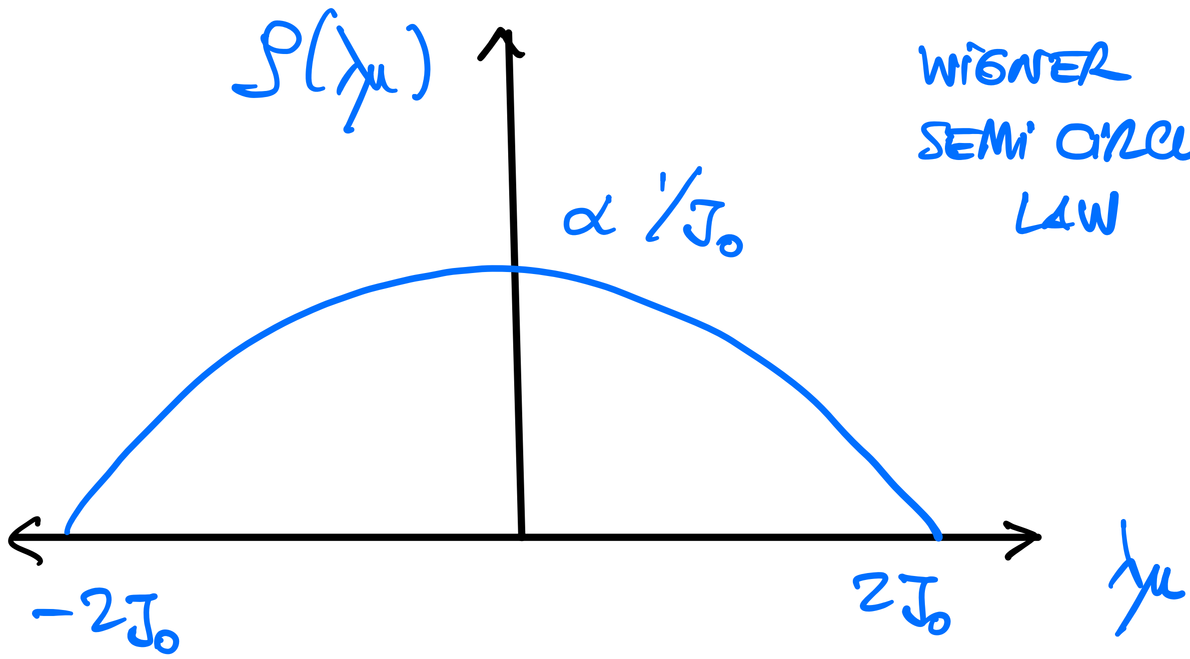
INTERESTED IN N FINITE AND SMALL

WE WANT TO STUDY $N \to \infty$

STATISTICAL MECHANICS PERSPECTIVE

IF THE $\{\lambda_\mu\}$ EIGENVALUES OF J_{ij} IN GOE

$\Longrightarrow$ SPHERICAL SK POT. ENERGY



WIGNER SEMI CIRCLE LAW

$$\rho(\lambda_\mu) = \frac{1}{2\pi J_0^2}\sqrt{4J_0^2 - \lambda_\mu^2}$$

KOSTERLITZ, THOULESS, JONES 70s

WHY CHOOSE THIS $\rho(\lambda_\mu)$: WE KNOW IT'S INTERESTING

RECALL WHAT WE WANT TO DO

- RECALL MODEL &

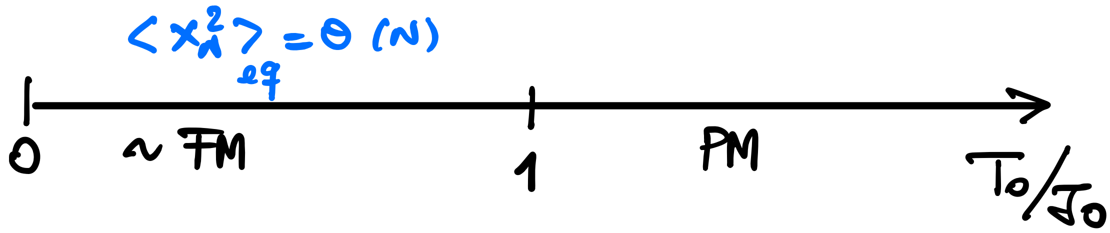
$$I_\mu = x_\mu^2 + \frac{1}{N} \sum_{\nu (\neq \mu)} \frac{x_\mu^2 p_\nu^2 + x_\nu^2 p_\mu^2 - 2 x_\mu p_\mu x_\nu p_\nu}{\lambda_\nu - \lambda_\mu}$$

- CHOOSE INITIAL COND
- CHOOSE QUENCH
- SOLVE DYN
- SOLVE GGE
- COMPARE $\langle x_\mu^2 \rangle$ & $\langle p_\mu^2 \rangle$ IN BOTH.

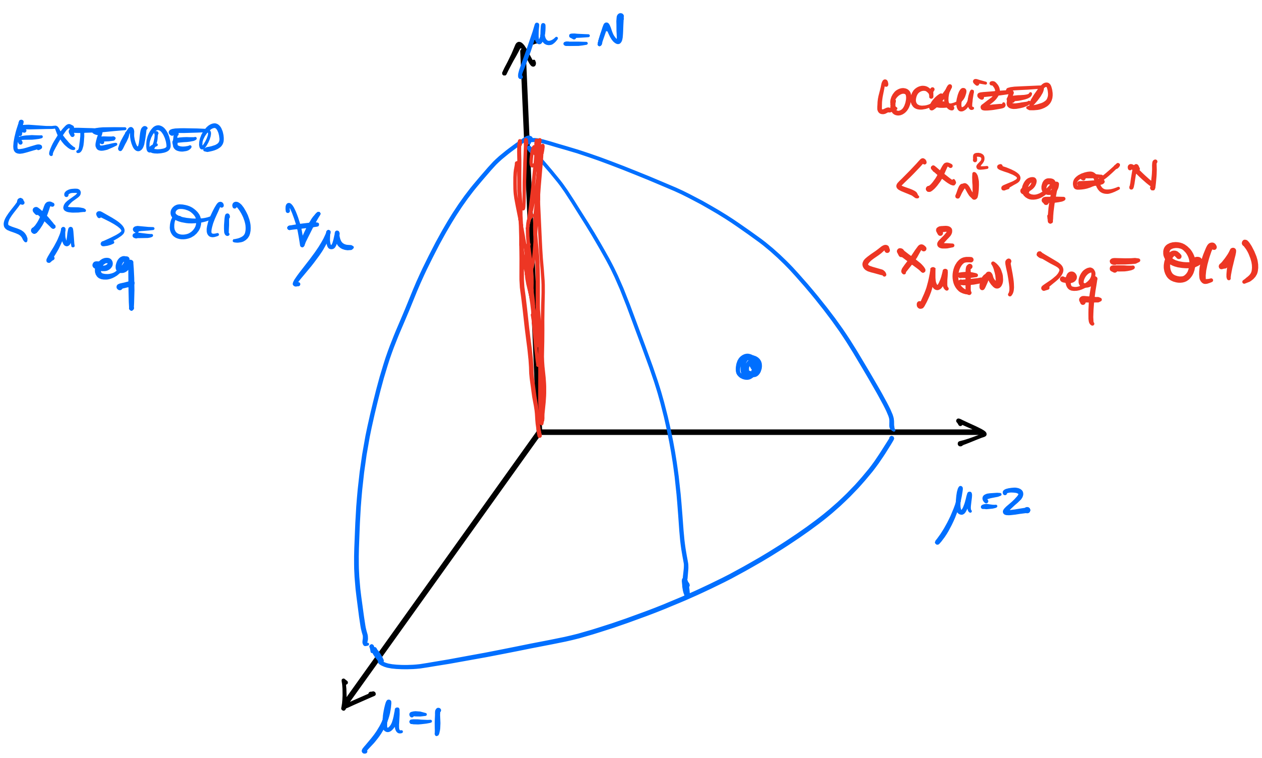
GIBBS-BOLTZMANN EQUILIBRIUM SSK

WE WILL DRAW THE $\underline{x}_0$ FROM $\rho_{GB} \propto e^{-\beta_0 H_0}$

$\exists$ PHASE TRANS. AT $\beta_0 J_0 = 1$



$x_N = \Theta(N^{1/2})$ LOCALIZED EXTENDED



WHY IS THE N-th MODE SPECIAL

$$H_{pot}(\vec{x}) = -\frac{1}{2}\sum_{\mu} \lambda_\mu^{(0)} x_\mu^2$$

NB WE CHOOSE A $-$ SIGN
HERE $(\lambda_\mu^{(0)} \geq 0)$

THE MINIMUM OF THE POT ENERGY IS ACHIEVED BY

$$x_\mu^{GS} = \sqrt{N}\,\delta_{\mu N}$$

WITH $E_{GS} = H_{pot}(\vec{x}^{\,GS}) = -\frac{\lambda_N^{(0)}}{2} N$

ONE CAN EXPECT THE N-th MODE TO BE

"MACROSCOPICALLY POPULATED"

AT LOW T_0

EQUILIBRIUM AT FINITE T_0

$$\langle x_\mu^2 \rangle_{eq} = \frac{T_0}{z_{eq} - \lambda_\mu^{(0)}}$$ IN THE INITIAL STATE

$$\sum_\mu \langle x_\mu^2 \rangle_{eq} \longrightarrow \int d\lambda\, \rho(\lambda) \langle x^2(\lambda) \rangle_{eq} = 1$$

FIXES $z_{eq} = \begin{cases} T_0 + J_0^2/T_0 & T_0 \geq J_0 \\ 2 J_0 & T_0 \leq J_0 \end{cases}$

SIMILARITY WITH PM-FM, BEC

KOSTERLITZ, THOULESS & JONES 70's
CRISANTI, SARRACINO & ZANNETTI ~2020

1 – THE MODEL HAS TWO EQUIL. STATES AT LOW T_0 RELATED BY $+ \leftrightarrow -$ SYMM. À LA FM

2 – THERE IS MACROSC. COND. ON A PARTICULAR "MODE" THE Nth ONE. À LA BEC

3 – THE Nth MODE $\mu = N$ IS PLAYING A SIMILAR ROLE AS THE $\vec{k} = 0$ ONE IN A FIELD THEORY (COARSE-GRAINED) DESCRIPTION OF PM-FM $\Longrightarrow$ ORDERING

MEANING OF QUENCH & ISOLATED EVOLUTION

- SYST H_0 COUPLED TO AN EQUIL. BATH AT T_0
  SINCE $t<0$ SUFFICIENTLY LONG SO THAT
  LET IT EQUILIBRATE WITH IT

- AT TIME $t=0$ COUPLING SWITCHED OFF.

- THEREAFTER NEWTON DYNAMICS
  if $H \neq H_0$ $\Longrightarrow$ NON-EQUIL EVOLUTION

  LONG-TIMES STATE ?

  STATISTICAL PHYSICS PROPERTIES ?

  SIMILAR TO EQUILIBRIUM IN SOME WAY ?

THE QUENCH

MANY OPTIONS TO CHANGE

$$H_0 \longrightarrow H$$

PROPOSE $\quad \lambda_\mu^{(0)} \longrightarrow \lambda_\mu = \lambda_\mu^{(0)} \frac{J}{J_0}$

RESCALING OF ALL SPRING CONSTANTS

$$x_\mu(0^+) = x_\mu(0^-) \qquad p_\mu(0^+) = p_\mu(0^-)$$

$$\Delta e_\mu = e_\mu(0^+) - e_\mu(0^-)$$

$$= \cancel{e_\mu^{kin}(0^+)} + e_\mu^{pot}(0^+) - \cancel{e_\mu^{kin}(0^-)} - e_\mu^{pot}(0^-)$$

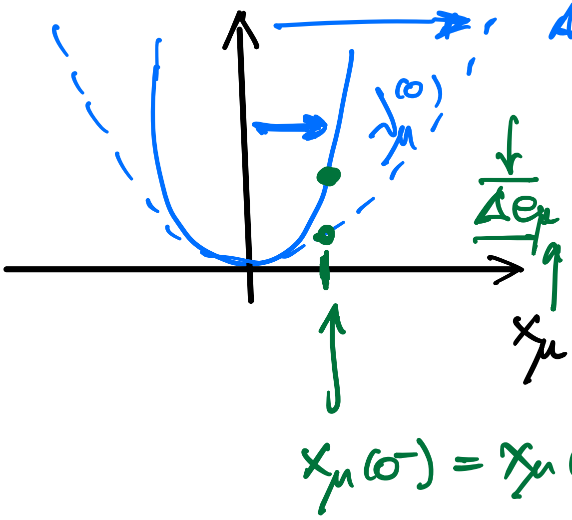
$$= -\frac{1}{2}\left(\lambda_\mu - \lambda_\mu^{(0)}\right) x_\mu^2(0)$$

$$= -\frac{1}{2} \underbrace{\left(\frac{J}{J_0} - 1\right)}_{\text{sign}} \underbrace{\lambda_\mu^{(0)}}_{\text{sign}} x_\mu^2(0)$$

e_N IS THE MOST AFFECTED ONE

$$\sum_\mu \Delta e_\mu = -\frac{1}{2}\left(\frac{J}{J_0} - 1\right) e_{pot}(0^-) \gtrless 0$$

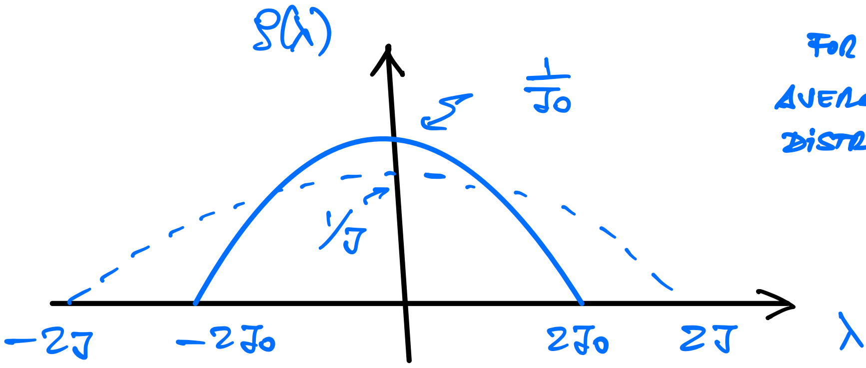
EXAMPLE WITH A PICTURE



AFTER THE QUENCH

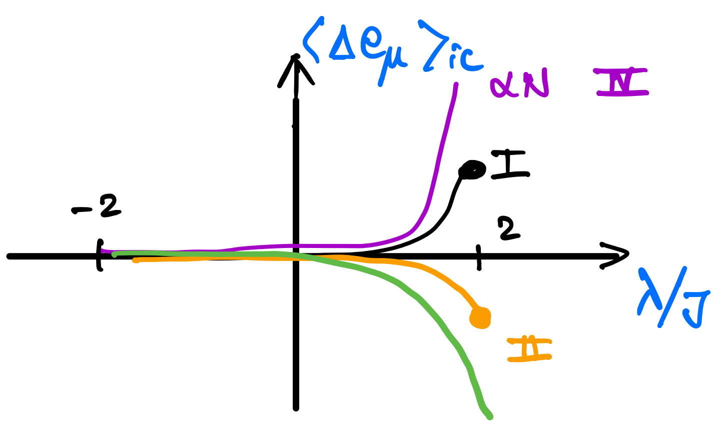
BEFORE THE QUENCH

e.g. $\lambda_\mu^{(0)} > 0$ HERE



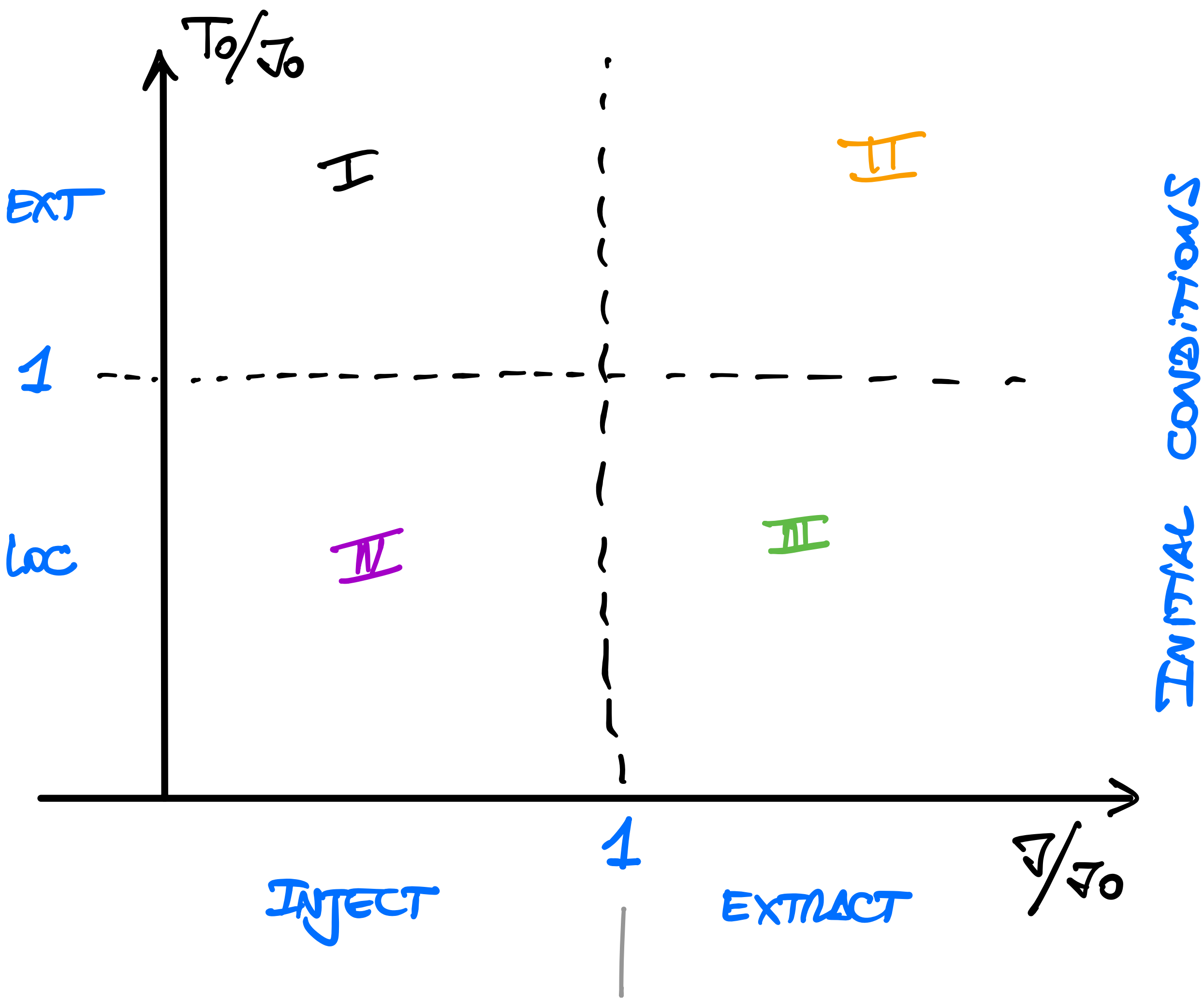
FOR THE AVERAGED DISTRIBUTION

FOR $J/J_0 > 1$



PARAMETER SPACE

CONTROL OF INITIAL COND.



TOTAL ENERGY VARIATION AT QUENCH

$\Delta E > 0$ $\quad$ $\Delta E < 0$

EQUIL. $J = J_0$

THE CONSTANTS OF MOTION

$\langle I_\mu \rangle_{ic}$ WE CAN COMPUTE THEM ON AVERAGE OVER $P(\psi_0)$

$$\langle I_\mu \rangle_{ic} = \begin{cases} a \dfrac{b-\lambda_\mu}{c-\lambda_\mu} & \forall \mu \quad T_0 > J_0 \\ a' \dfrac{b'-\lambda_\mu}{c'-\lambda_\mu} & \mu \neq N \quad T_0 < J_0 \\ \left(1-T_0/J_0\right)\left(1-\dfrac{T_0}{J}\right) N + O(1) & \mu = N \end{cases}$$

a, b, c, a', b', c' DEPEND ON $\left(\frac{T_0}{J_0}, \frac{J}{J_0}\right)$

NB. SPECIAL SCALING $\langle I_N \rangle_{ic}$ FOR $T_0 < J_0$

HOW TO SOLVE THE DYNAMICS ?

IT'S AN ALMOST QUADRATIC MODEL.
THE COORD. ARE COUPLED BY THE LAGRANGE MULT.
THE HAMILT & THE $\{I_\mu\}$s ARE IN FACT QUARTIC.

BUT LARGE N LIMIT HELPS

1 – METHODS OF DISORDERED SYSTEMS

YIELD

SCHWINGER-DYSON EQS FOR

$$C(t,t') = \frac{1}{N} \sum_\mu \left[\langle x_\mu(t)\, x_\mu(t') \rangle_{ic} \right]$$

$$R(t,t') = \frac{1}{N} \sum_\mu \left[\left. \frac{\delta \langle x_\mu(t) \rangle_{ic}^{(h)}}{\delta h(t')} \right|_{h=0} \right]$$

WHERE $[\cdots] = \int d\lambda\, \rho(\lambda) \cdots$

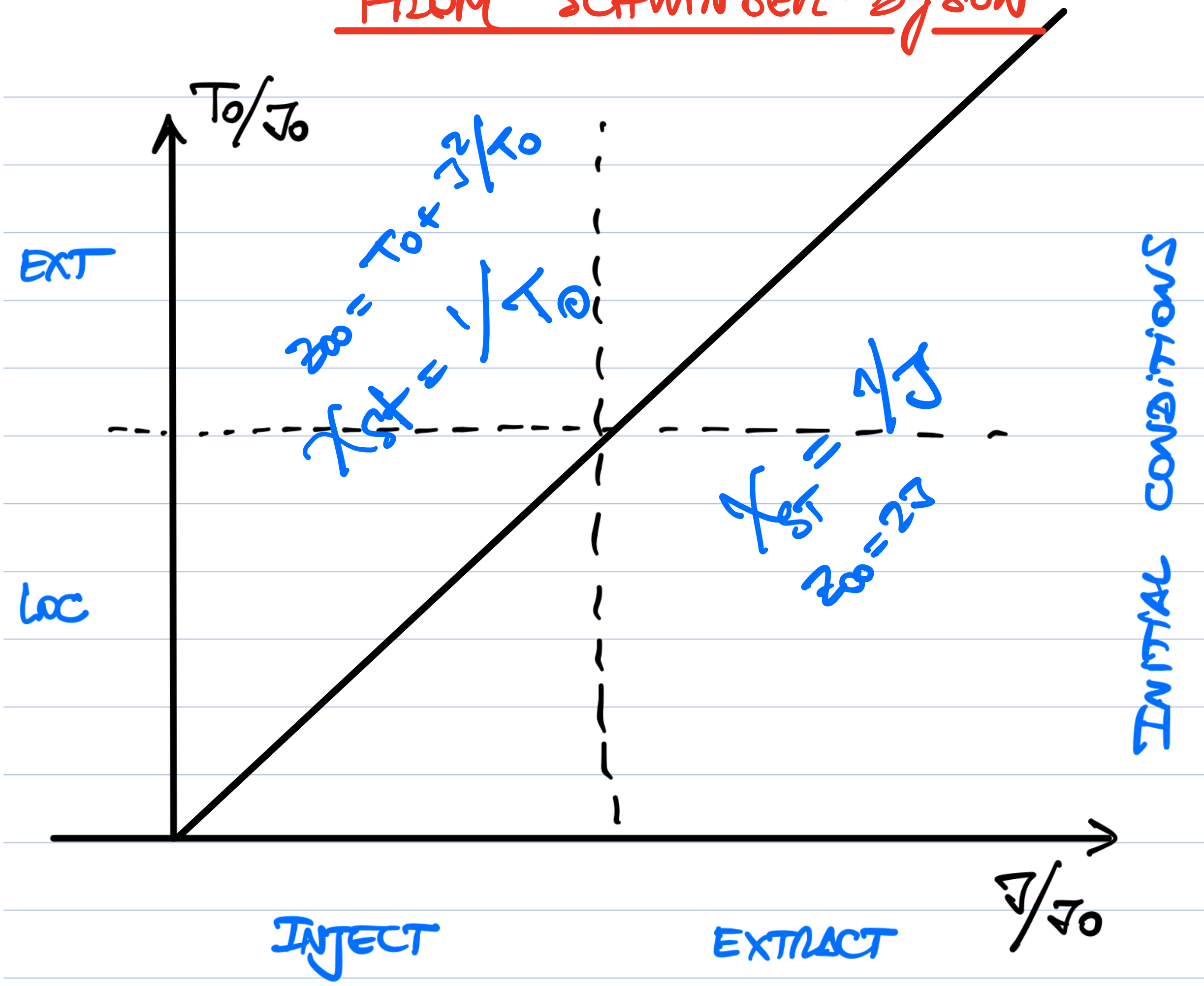
AVERAGE OVER THE DIST OF SPRING CONST.

EXACT FOR $N \to \infty$

& SOLVABLE NUMERICALLY W/ SOME ANALYTIC RESULTS AS WELL.

FIRST OBSERVATION: STEADY STATE $\forall$ PARAM.

DYNAMIC PHASE DIAGRAM FROM SCHWINGER-DYSON



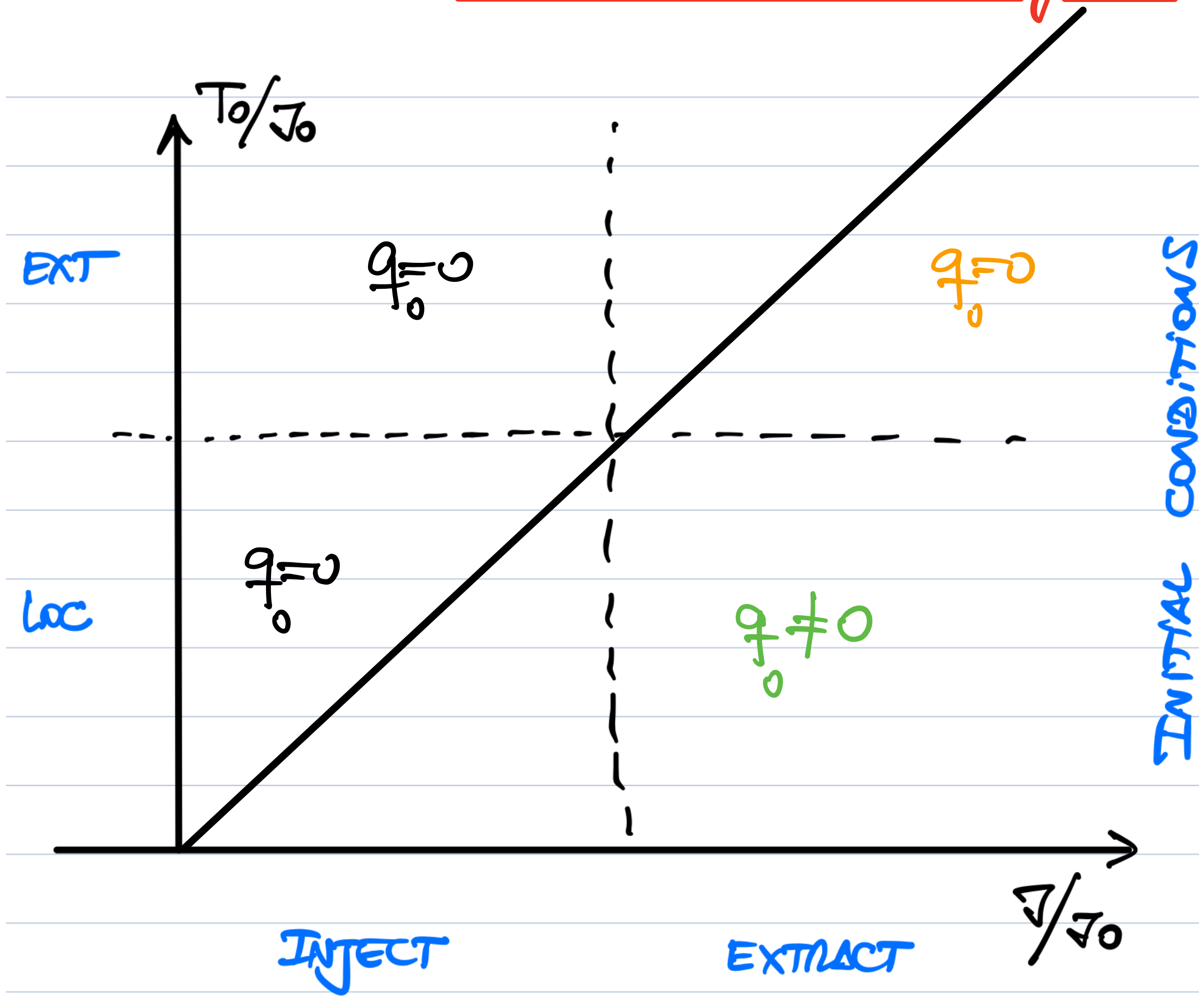
TOTAL ENERGY VARIATION AT QUENCH

DISTINGUISHED BY $z_\infty = \lim_{t \to \infty} z(t)$

$$\chi_{st} = \lim_{t\to\infty} \int_0^t dt' \, R(t')$$

$\hookrightarrow \begin{cases} T_0 + J^2/T_0 \\ 2J \end{cases}$

DYNAMIC PHASE DIAGRAM FROM SCHWINGER-DYSON



TOTAL ENERGY VARIATION AT QUENCH

DISTINGUISHED BY $q_0 = \lim_{t\to\infty} C(t,0)$

(ALSO $q = \lim_{t\to\infty} \lim_{t_w\to\infty} C(t,t_w)$)

2 – METHODS OF PARAMETRIC OSCILLATORS

SOTIRIADIS & CARDY '10s
+ OLDER ONES

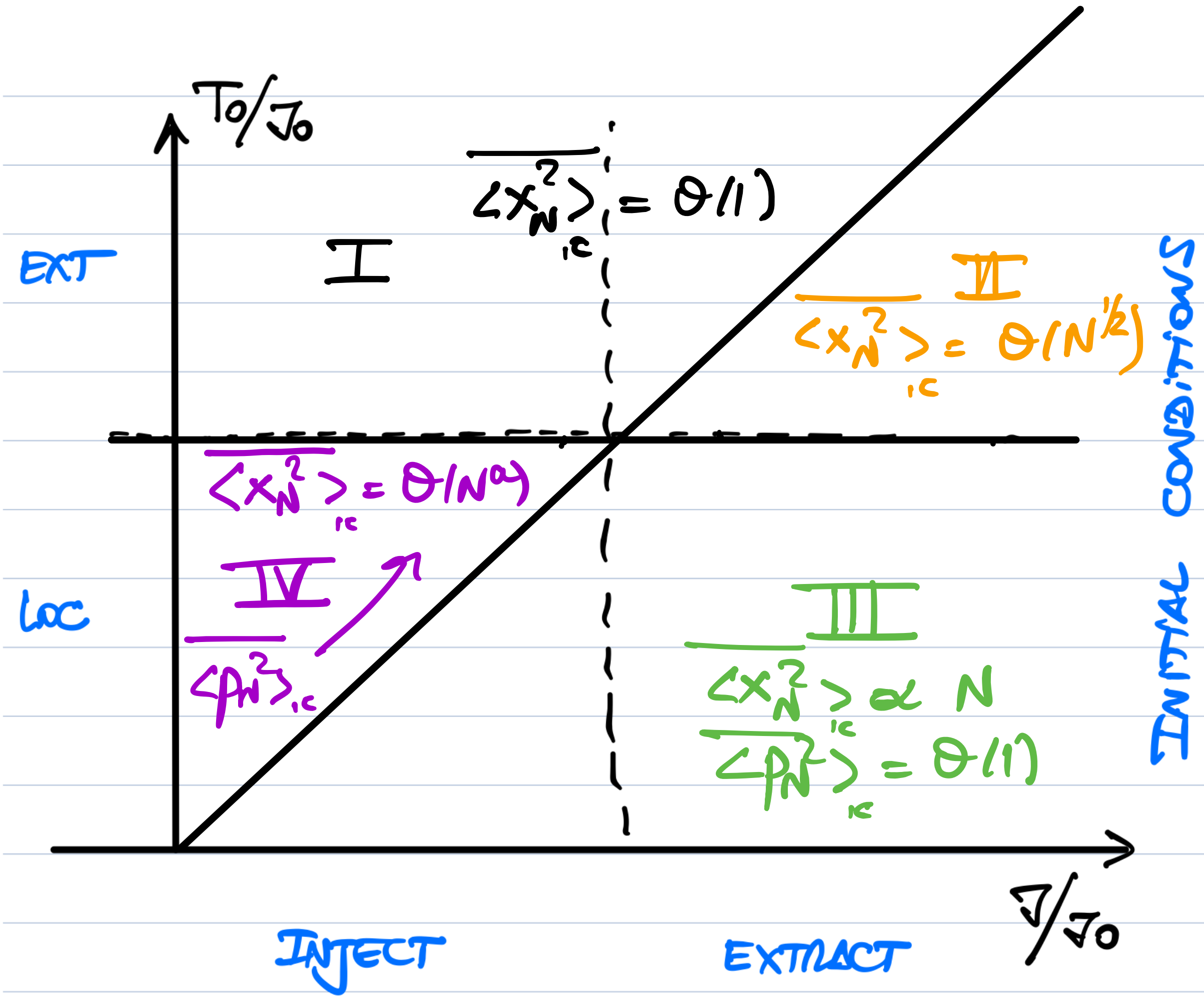
YIELD

$$\langle x_\mu^2(t) \rangle_{iC} \qquad \langle p_\mu^2(t) \rangle_{iC}$$

FOR N FINITE (A NUMERICAL EVALUATION IS NEEDED $\Longrightarrow$ (t, N) RESTRICTED)

THEN AVERAGE OVER A TIME WINDOW

DYNAMIC PHASE DIAGRAM FROM FINITE N ANALYSIS



TOTAL ENERGY VARIATION AT QUENCH

ANALYSIS OF

$\overline{\langle x_N^2 \rangle}_{ic}$ & $\overline{\langle p_N^2 \rangle}_{ic}$

WHAT IS THE PARTICLE DOING ?

IN SECTORS I AND II
THE PARTICLES MOVE AROUND THE WHOLE SPHERE

II IS REMINISCENT OF THE USUAL LOW T QUENCHES (PECULIAR SCALING W/N)

IN SECTORS III & IV THERE IS "LOCALIZATION" ON THE Nth MODE

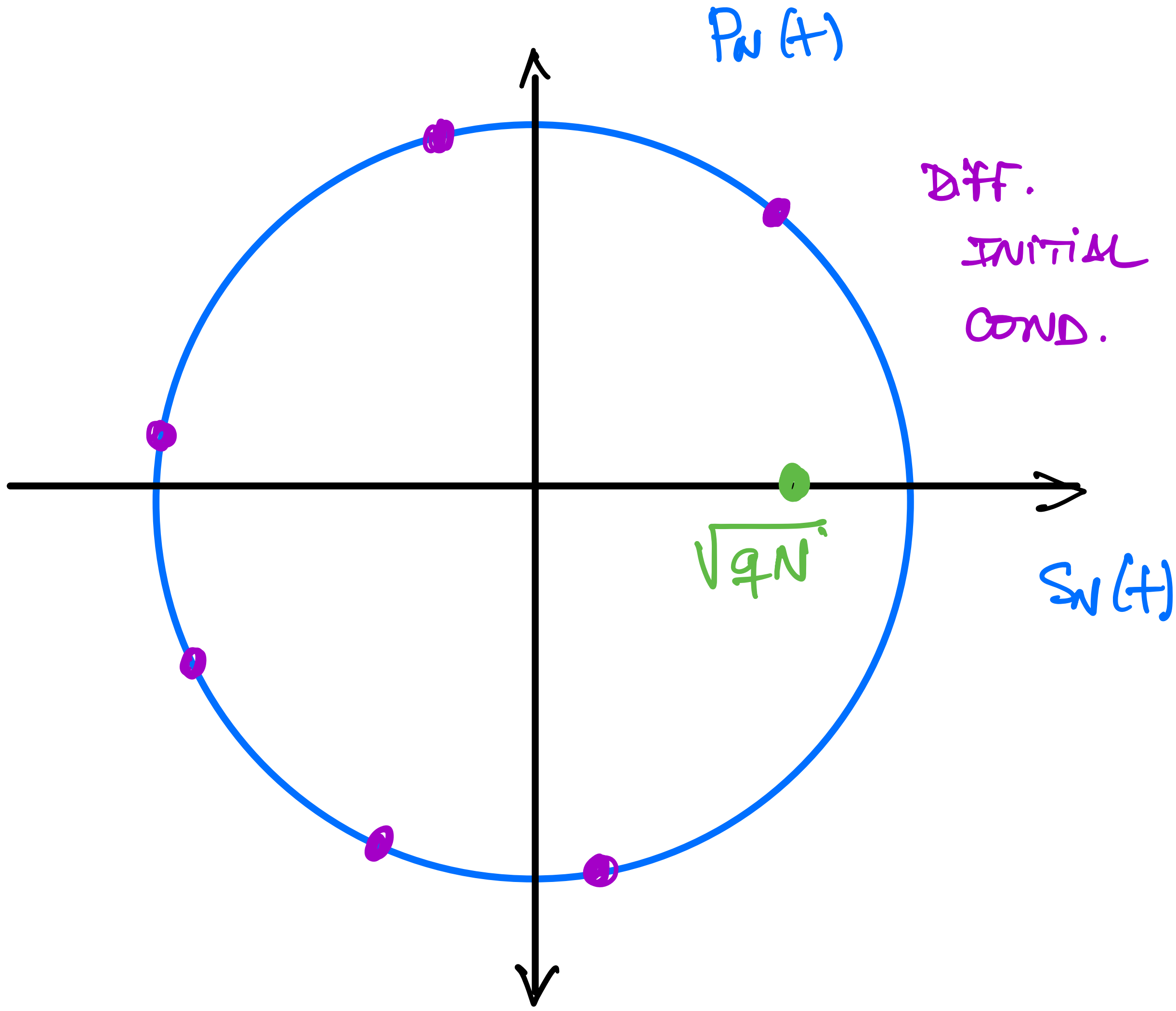
- EITHER STAYS CLOSE TO ITS "ORIGINAL" POSITION III
- OSCILLATIONS WITH $\Theta(N)$ AMPLITUDE IV

SUBTLETIES ABOUT IMPOSING CONSTRAINT STRICTLY OR ON AVERAGE NOT DISCUSSED

KAC & THOMPSON 50s ZANNETTI
RECENT YEARS

RELATED TO INEQUIVALENCE OF ENSEMBLES

WHAT IS THE PARTICLE DOING ?



THE Nth DIRECTION IN THE CONDENSED SECTORS III & IV

STATIONARY DYNAMICS IN STEADY STATE

IS IT GIBBS - BOLTZMANN ?

$$C(t,t') \longrightarrow C_{st}(t-t')$$

$$R(t,t') \longrightarrow R_{st}(t-t')$$

FDT CHECKS.

IN EQUIL. AT A TEMP β_f

$$R_{st}(t-t') = -\frac{1}{T_f}\, d_t\, C_{st}(t-t') \qquad \text{FDT}$$

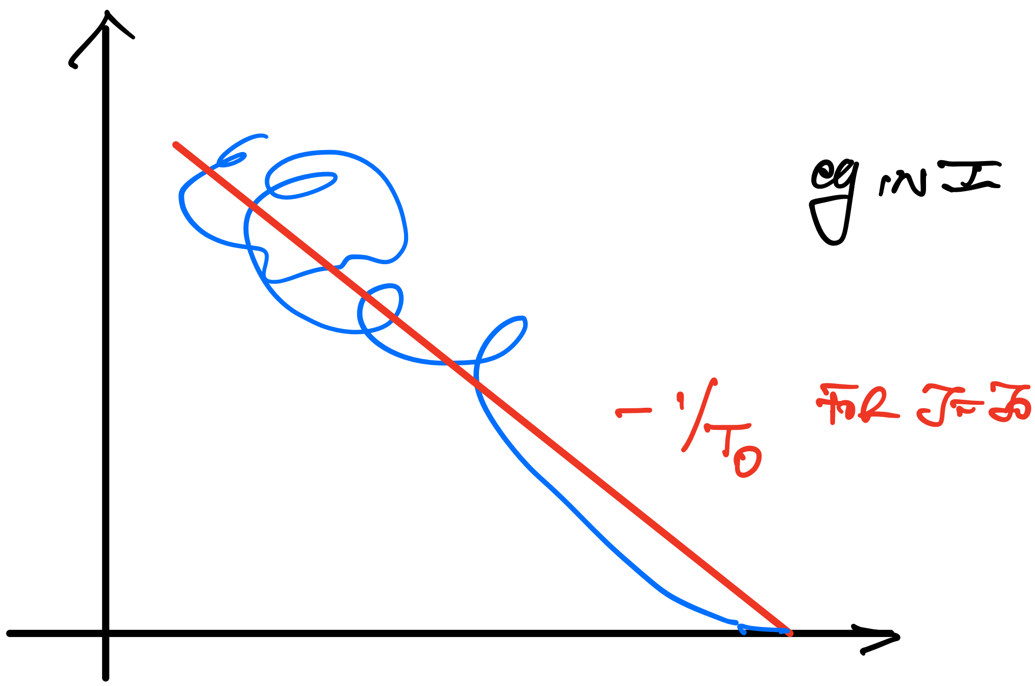
$$\chi_{st}(\tau) = \int_0^\tau d\tau'\, R_{st}(\tau') = \frac{1}{T_f}\left[1 - C_{st}(\tau)\right]$$

LINEAR RELATION BTW INTEG. LINEAR RESP & CORR FCT

PLOT $\chi_{st}(\tau)$ VS $C_{st}(\tau)$ WITH τ AS A PARAM

VIOLATIONS OF FDT

SHOW PLOT FROM SLIDES OR
MAKE SKETCH



FOR ALL SETS OF PARAMETERS, APART
FROM $J = J_0$ AND

$$\left(\frac{T_0}{J_0}\right)^2 = \frac{J}{J_0} \quad \text{AND } T_0 > J_0$$

WHERE WE SEE A T_f = CONST.
THOUGH THERE'S ENERGY EXTRACTION.

THE GGE

SPECIAL CASE $J = J_0$ EQUILIBRIUM

$$\sum_\mu \gamma_\mu I_\mu$$

IF WE CHOOSE $\gamma_\mu = -\frac{\lambda_\mu \beta}{2}$

$$\sum_\mu \gamma_\mu I_\mu = -\frac{\beta}{2} \sum_\mu \lambda_\mu I_\mu = \beta H$$

AND THE GIBBS-BOLTZMANN MEASURE IS RECOVERED.

HOW TO MANIPULATE & CALCULATE AVERAGES WITH THE GGE ?

DISORDERED SYST METHODS

HOW TO MANIPULATE THE GGE ?

$$Z_{GGE} = \int d^N x \, d^N p \; e^{-\sum_\mu \gamma_\mu I_\mu}$$

WITH γ_μ FIXED FROM $\langle I_\mu \rangle_{ic} = \langle I_\mu \rangle_{GGE}$

N- EQS.

$$I_\mu = x_\mu^2 + \frac{1}{mN} \sum_{\nu(\neq\mu)} \frac{(x_\mu p_\nu - x_\nu p_\mu)^2}{\lambda_\nu - \lambda_\mu}$$

IN Z_{GGE} QUARTIC EXPRESSION IN INTEG. VARIABLES IN THE EXPONENTIAL

$\Rightarrow$ NO OBVIOUS GAUSSIAN INT.

"HUBBARD- STRATONOVICH" KIND OF DECOUPLINGS

$$A_\mu^{(x)} = \sum_{\nu(\neq\mu)} \frac{p_\nu^2}{\lambda_\nu - \lambda_\mu}$$

$$A_\mu^{(p)} = \sum_{\nu(\neq\mu)} \frac{x_\nu^2}{\lambda_\nu - \lambda_\mu}$$

etc.

THE EXPRESSION IN THE EXPONENTIAL BECOMES

$$N\, S\left[x_\mu^2,\ p_\mu^2,\ A_\mu^{(x)},\ A_\mu^{(p)},\ x_\mu p_\mu,\ A_\mu^{(px)},\ z\right]$$

TEMPTING TO EVALUATE BY SADDLE-POINT

PBM: OVER $O(N)$ VARIABLES ?

SOL: GO TO LARGE N LIMIT

$$\frac{1}{N}\sum_\mu \cdot \longrightarrow \int d\lambda\, \rho(\lambda)\ \cdot$$

SADDLE-POINT ON FUNCTIONS (AS IN GINZBURG-LANDAU)

$\Longrightarrow$ SADDLE POINT EQS ON

$$x^2(\lambda),\ p^2(\lambda),\ A^{(x)}(\lambda),\ A^{(p)}(\lambda),\ z$$

CROSSED xp CAN BE SHOWN TO VANISH

(ALSO, NO NEED TO IMPOSE SECOND. CONST $\vec{x}\cdot\vec{p}=0$)

AS USUAL

$$\begin{cases} x_{SP}^2(\lambda) = \langle x^2(\lambda) \rangle_{GGE} \\ p_{SP}^2(\lambda) = \langle p^2(\lambda) \rangle_{GGE} \end{cases}$$

ANSATZ $\langle x^2(\lambda) \rangle_{GGE} = \dfrac{T(\lambda)}{z - \lambda}$

$$\langle p^2(\lambda) \rangle_{GGE} = T(\lambda)$$

WITH THE POSSIBILITY OF NEEDING TO SEPARATE THE Nth MODE AND TREAT IT SEPARATELY WITH $O(N)$ SCALING IN $\boxed{\text{III}}$ AND $\boxed{\text{IV}}$

WE SHOW THAT THEY SOLVE THE S.P. EQS.

QUESTION NOW, HOW TO FIX $T(\lambda)$?

IMPOSING $\langle I_\mu \rangle_{iC} = \langle I_\mu \rangle_{GGE}$

RECALL $\langle I(\lambda) \rangle_{iC}$ ARE KNOWN

WHAT ABOUT rhs ?

$$\langle I(\lambda) \rangle_{GGE} = 2 \langle x^2(\lambda) \rangle_{GGE} \left\{ 1 + \frac{1}{m} \int d\lambda' \frac{\langle p^2(\lambda') \rangle_{GGE}}{\lambda' - \lambda} - \delta_{\mu N} \frac{\langle x_N^2 \rangle}{2N} \right\}$$

WE REPLACE HERE THE ANSATZ FOR $\langle x^2(\lambda) \rangle_{GGE}$ AND $\langle p^2(\lambda) \rangle_{GGE}$ AND

YIELD THE EQS THAT FIX $T(\lambda)$

HOW TO SOLVE THE EQ. ON $T(\lambda)$

- NUMERICALLY
- ALSO ANALYTICALLY!

GO TO COMPLEX PLANE

IDENTIFY $\rho(\lambda) T(\lambda) = \operatorname{Im} X(\lambda)$

MANIPULATE THE EQ W/ TRICKS OF RANDOM MATRIX THEORY

THANKS TO VOLODYA KAZAKOV

FIND A QUADRATIC EQ FOR

$$\left(\rho(\lambda)\, T(\lambda)\right)^2$$

$\Rightarrow$ SOLVE IT $\Rightarrow$ $T(\lambda)$

& IN SECTION IV ALSO T_N

WITH $\{T(\lambda), T_N\}$ WE KNOW

$$\langle x^2(\lambda)\rangle_{GGE} \qquad \langle p^2(\lambda)\rangle_{GGE}$$

COMPARE TO THE DYN RESULTS

MODE RESOLVED, ONLY FOR N FINITE

VERIFY

$$\lim_{t\to\infty} \lim_{N\to\infty} \overline{\langle A(t)\rangle_{IC}}$$

$$= \langle A \rangle_{GGE}$$

WITH $A = x_\mu^2$ $\quad B = p_\mu^2$ $\quad \forall \mu$.

- WE HAVE ANOTHER SET OF EQS FOR $\{\gamma_\mu\} \to \{\gamma(\lambda)\}$
- $\sum_\mu \gamma_\mu I_\mu \longrightarrow \sum_\mu \beta_\mu H_\mu^{(2)}$

WHICH YIELDS THE CORRECT AVERAGES.

THE SPECIAL CURVE

$$\left(\frac{I_0}{J_0}\right)^2 = \frac{J}{J_0} \qquad \text{FOR} \quad \frac{I_0}{J_0} > 1$$

THE DYN. "EQUILIBRATES" IN THE SENSE OF THERE BEING A SINGLE TEMP. $T_\mu \longrightarrow T_f$

SPECIAL ON THE CURVE $\langle I_\mu \rangle_{IC} = I$
ALL ARE EQUAL

PROVEN WITHIN GGE & ALSO SEEN WITH THE SCHWINGER-DYSON EQ.

- IS IT A COINCIDENCE?
- ARE THERE ALWAYS A SET OF PARAM FOR WHICH THERE'S A SINGLE T_f?

DON'T KNOW

IS THE GLOBAL FDT RELATION TELLING US SOMETHING?

$$R(t,t') = \frac{\delta \langle x(t) \rangle^h}{\delta h(t')}$$

$\langle \cdots \rangle$ over dyn
eg. i.c.

$$C(t,t') = \langle x(t)\, x(t') \rangle$$

SINGLE VARIABLE NOTATION, FOR SIMPLICITY

STAT LIMIT

$$R(t,t') \longrightarrow R_{st}(t-t')$$
$$C(t,t') \longrightarrow C_{st}(t-t')$$

IF USUAL BOLTZM EQ $\quad R_{st}(t-t') = -\frac{1}{T} \frac{d C_{st}(t-t')}{d(t-t')}$

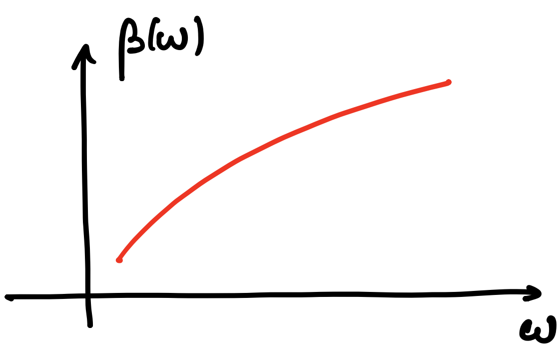
FOR $t-t' \geq 0$

FOURIER TRANSFORM

$$\mathrm{Im}\, \tilde{R}_{st}(\omega) = \frac{\beta\, \omega\, \tilde{C}_{st}(\omega)}{2}$$

FACTOR $1/2$ & SIGN DEPEND ON $\tilde{C}_{st}(\omega)$ DEF.

$$\frac{2\, \mathrm{Im}\, \tilde{R}(\omega)}{\omega\, \tilde{C}(\omega)} = \beta(\omega) \quad \text{OUT OF EQUIL.}$$



e.g. IN SECTOR I
ANOTHER WAY TO
STATE OUT OF
BOLTZMANN EQ.

IS $\beta(\omega)$ RELATED TO $\beta(\lambda)$ FROM THE GGE ?

YES. IDEA ONE-TO-ONE RELATION BETWEEN
ω AND λ
"LIKE AN INDEX RELATION"

ex. HARM OSC $H = \frac{p^2}{2m} + \frac{m\omega_0^2}{2} x^2$

$$R(\tau) = \frac{1}{m\omega_0} \sin \omega_0 \tau \quad \theta(\tau)$$

$$\overline{C(\tau)} = \frac{1}{2} \left(x_0^2 + \frac{p_0^2}{m_0^2 \omega_0^2} \right) \cos \omega_0 \tau$$

WHERE $\overline{\cdots}$ AVERAGE OVER TIME - WINDOW
AROUND t' TO KILL NOT STAT TERM

THE FDT- RELATION

$$\mathrm{Im}\, \tilde{R}(\omega) = \frac{\pi}{2m\omega_0} \delta(\omega - \omega_0)$$

$$\omega \tilde{C}(\omega) = \frac{\pi \omega}{2m\omega_0^2} \underbrace{\left(m\omega_0^2 x_0^2 + \frac{p_0^2}{m} \right)}_{2E_0} \delta(\omega - \omega_0)$$

$$\beta_{\mathrm{eff}}(\omega) = \frac{2\, \mathrm{Im}\, \tilde{R}(\omega)}{\omega \tilde{C}(\omega)} = \frac{1}{E_0} \qquad \text{FOR } \omega = \omega_0$$

THE GGE

$$\rho_{GGE} \propto e^{-\beta_{GGE} H}$$

β_{GGE} FROM $\langle H \rangle_{GGE} = E_0 \implies$

$$\beta_{GGE} = \frac{1}{E_0}$$

$$\implies \qquad \beta_{GGE} = \beta_{eff}(\omega_0) = 1/E_0$$

MANY OSC. WITH FREQ $\omega_0^{(k)} \implies$

$$\beta_{GGE}^{(k)} = \beta_{eff}(\omega_0^{(k)})$$

- SO THE GGE MODES $\iff$ FDT MEASUREMENT.
- NO CONTRADICTION W/ T_{eff} IDEAS, IN THE LONG RUN THE MODES OF THE NEUMANN BECOME INDEP. TOO

DEVELOP. OF THESE IDEAS

DE NARDIS, FOINI, PANFIL, GAMBASSI

KONIK & LFC

CONCLUSIONS

- WE SOLVED THE OUT OF EQ DYN OF THE NEUMANN MODEL

  W/ SCHWINGER DYSON & MODE-BY-MODE

- WE CALCULATED AVERAGES W/ THE GGE

  THEY COINCIDE!

NON-TRIVIAL PROBLEM. W/ PHASE TRANSITIONS