Quantum effects in nonequilibrium systems with slow dynamics*

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1 Introduction

Examples of classical systems that slowly evolve towards equilibrium though without reaching it in observable times scales are spin-glasses [1, 2, 3], vortex glasses [4, 5] and all other type of glasses [1, 6], systems undergoing domain growth [2, 5], etc. Besides, systems that would reach equilibrium in observable time-scales like complex liquids can be driven out of equilibrium by external perturbations [1, 2, 8, 9]. Powders stay in static metastable states unless externally perturbed by tapping, vibration or a shearing that drives them out of equilibrium and makes them slowly evolve towards more compact configurations [10]. The non-equilibrium dynamics of these systems has been studied with a theoretical approach that is based on the solution of mean-field simple models [1, 11]. Even if *a priori* very different, many common features have been identified. They lead to a common picture for the dynamics of classical systems in the limit of small entropy production [1, 12] where the effective temperature [13] is crucial.

In several cases of practical interest, quantum effects play an important role. For instance, glassy phases at very low temperatures have been identified in a large variety of materials (spin-glass like systems, interacting electrons with disorder, materials undergoing super-conductor transitions, glasses, etc. [14]). Clearly, the driven case is also very important in systems with quantum fluctuations, take for instance an interacting electronic system driven by an external current. It is then interesting to study if the approach developed and the results obtained for the classical

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dynamics in the limit of small entropy production carries through when quantum fluctuations are included.

The basic techniques used to study classical glassy models with or without disorder are well documented in the literature (the replica trick, scaling arguments and droplet theories, the dynamic functional method used to derive macroscopic equations from the microscopic Langevin dynamics, functional renormalization, Montecarlo and molecular dynamic numerical methods). On the contrary, the techniques needed to deal with the statics and dynamics of quantum macroscopic systems are much less known in general. I shall start by briefly discussing the role played by the environment in a quantum system and by introducing and comparing the equilibrium and dynamic approaches. Recently, a detailed presentation appeared in [16]. Next, I shall discuss the role played by explicit quenched disorder in the calculations. At the end, I shall describe the behaviour of a toy glassy model with quantum fluctuations [15],[17]-[19] and I shall confront it to the picture derived in the classical limit.

2 The role played by the environment

In the typical experimental protocol used to study a system in its glassy phase one "quenches" (infinitely rapid descent in temperature from the disordered to the ordered phase) or "anneals" (one tunes the external temperature with a finite cooling rate) the sample at an initial time and subsequently follows its evolution in time [1, 2, 3, 20]. Once arrived at the final point in parameter space, if no external forces are applied [21], the system relaxes; its energy density decreases towards an asymptotic value. Hence, the system is not isolated but in contact with a thermal environment that acts as a source of dissipation. The system plus environment is "closed" while the system alone is "open". The nature of the environment, *e.g.* whether it can be modeled by a classical or a quantum ensemble, will depend on the problem under study. Right after a quench or a rapid annealing, the system is not in equilibrium with the reservoir. Whether it will eventually reach equilibrium in experimentally relevant time scales depends on the nature of the system and its interaction with the environment. Indeed, for glassy systems at low temperatures the equilibration time scales becomes so long that it crosses over the experimentally observable times.

We shall then study the coupled system

$$H_{\text{TOT}} = H_{\text{SYST}} + H_{\text{ENV}} + H_{\text{INT}} + H_{\text{COUNTER}} \; . \tag{1}$$

For simplicity we shall focus on a single particle coupled to an environment. H_{SYST} is then the Hamiltonian of the isolated particle, $H_{\text{SYST}} = p^2/(2M) + V(q)$, with p and q its momentum and position. H_{ENV} is the Hamiltonian of a thermal bath that, again for simplicity, we take to be an ensemble of N independent Harmonic oscillators

$$H_{\text{ENV}} = \sum_{a=1}^{N_b} \frac{\pi_a^2}{2m_a} + \frac{m_a \omega_a^2}{2} x_a^2 \; . \tag{2}$$

This is indeed a very usual choice since, for example, it may represent phonons. H_{INT} is the coupling between system and environment. We shall restrict the following discussion to an interaction that is linear in the oscillator coordinates, x_a, and in the particle coordinate, $H_{\text{INT}} = q\sum_{a=1}^{N_b} c_a x_a$, with c_a the coupling constants. The calculations can be easily generalized to an interaction with a more complicated dependence on the system's coordinate, $F(q)$, that may be dictated by the symmetries of the system. We shall discuss the last term, H_{COUNTER}, below.

2.1 Static properties

In order to analyze the statistical static properties of the coupled system, we shall study the Gibbs functional, Z_{TOT}, in the classical limit and the density operator, $\hat{\rho}_{\text{TOT}}$, when quantum fluctuations are present.

2.1.1 Classical problems

The Gibbs functional of the total system reads

$$Z_{\text{TOT}}[\eta] = \sum_{\substack{\text{CONF OSC} \\ \text{CONF SYST}}} \exp(-\beta H_{\text{TOT}} - \beta\eta q) \tag{3}$$

where the sum represents and integration over the phase space of the full system, the particle's and the oscillators', and η is a source. Having chosen a quadratic bath and a linear coupling, the integration over the oscillators' coordinates and momenta can be easily performed. This yields the reduced Gibbs functional

$$Z_{\text{RED}}[\eta] = \sum_{\text{CONF SYST}} \exp\left[-\beta\left(H_{\text{SYST}} + H_{\text{COUNTER}} + \eta q - \frac{1}{2}\sum_{a=1}^{N_b}\frac{c_a^2\omega_a^2}{m_a}\, q^2\right)\right] . \tag{4}$$

The "counterterm" H_{COUNTER} is chosen to cancel the last term in the exponential and it avoids the renormalization of the particle's mass due to the coupling to the environment [22]. Z_{RED} coincides with the partition function of the particle and its equilibrium properties are hence recovered. The interaction with the reservoir does not modify the statistical properties of the particle.

2.1.2 Quantum problems

The quantization rules are imposed to the system and the bath variables,

$$[\hat{p},\hat{q}] = -i\hbar \qquad [\hat{\pi}_a, \hat{x}_b] = -i\hbar\delta_{ab} \ . \tag{5}$$

The equilibrium density matrix reads

$$\begin{aligned} \rho_{\text{TOT}}(q'', x''_a; q', x'_a) &= \langle q'', x''_a|\, \hat{\rho}_{\text{TOT}}\, |q', x'_a\rangle \\ &= \frac{1}{Z_{\text{TOT}}}\langle q'', x''_a|\ \exp(-\beta\hat{H}_{\text{TOT}})\ |q', x'_a\rangle \ , \end{aligned} \tag{6}$$

with the partition function given by $Z_{\rm TOT} = {\rm Tr}\exp(-\beta\hat{H})$ and the trace taken over all the states of the full system. As usual [23, 24], the density matrix can be represented by a functional integral in imaginary time,

$$\rho_{\rm TOT}(q'', x''_a; q', x'_a; \eta) = \frac{1}{Z_{\rm TOT}} \int_{q(0)=q'}^{q(\hbar\beta)=q''} \mathcal{D}q(\tau) \int_{x_a(0)=x'_a}^{x_a(\hbar\beta)=x''_a} \mathcal{D}x_a(\tau) \exp\left[-\frac{1}{\hbar} S^{\rm E}_{\rm TOT}[\eta]\right] . \tag{7}$$

The functions $q(\tau)$ and $x_a(\tau)$ run over the imaginary time τ, that varies between 0 and $\hbar\beta$. The Euclidean action $S^{\rm E}$ has contributions from the system, the reservoir, the interaction and the counterterm:

$$S^{\rm E}_{\rm TOT}[\eta] = \int_0^{\beta\hbar} d\tau \left(\mathcal{L}^{\rm E}_{\rm SYST} + \mathcal{L}^{\rm E}_{\rm ENV} + \mathcal{L}^{\rm E}_{\rm INT} + \mathcal{L}^{\rm E}_{\rm COUNTER} + \eta(\tau)q(\tau)\right) , \tag{8}$$

and we have included a source η coupled to the particle's coordinate in order to use it to compute expectation values:

$$\langle A(q)\rangle = {\rm Tr}\; A(\hat{q})\hat{\rho}_{\rm TOT} . \tag{9}$$

The interaction term reads

$$\mathcal{L}^{\rm E}_{\rm INT} = \int_0^{\hbar\beta} d\tau\, q(\tau) \sum_{a=1}^{N_b} c_a x_a(\tau) . \tag{10}$$

As in the classical case, the path integral over the oscillators' coordinates and positions is quadratic. The calculation of expectation values involves a trace over states of the full system. For operators that depend only on the particle, as $A(\hat{q})$ above, the trace over the oscillators can be done explictly. Hence, if one constructs the reduced density operator ${\rm Tr}_{\rm OSC}\; \hat{\rho}_{\rm TOT}$ that acts on the system's Hilbert space, the expectation value of the observables of the system is given by

$$\langle A(q)\rangle = {\rm Tr}_{\rm SYST}\; A(\hat{q})\hat{\rho}_{\rm RED} . \tag{11}$$

The reduced density matrix reads

$$\begin{aligned} \rho_{\rm RED}(q'', q') &= {\rm Tr}_{\rm OSC}\rho_{\rm TOT}(q'', x''_a; q', x'_a) \\ &= \frac{1}{Z_{\rm RED}} \int_{q(0)=q'}^{q(\hbar\beta)=q''} \mathcal{D}q(\tau) \exp\left[-\frac{1}{\hbar}\left(S^{\rm E}_{\rm SYST} + S^{\rm E}_{\rm COUNTER} - \frac{1}{2}\sum_{a=1}^{N_b} \frac{c_a^2\omega_a^2}{m_a} \int_0^{\hbar\beta} d\tau q(\tau)^2 \right.\right. \\ &\qquad \left.\left. + \int_0^{\hbar\beta} d\tau \int_0^{\tau} d\tau'\; q(\tau)K(\tau-\tau')q(\tau')\right)\right] \end{aligned} \tag{12}$$

where $Z_{\rm RED}$ is the partitition function of the reduced system, $Z_{\rm RED} = Z_{\rm TOT}/Z_{\rm OSC}$ and $Z_{\rm OSC}$ the partition function of isolated ensemble of oscillators. The interaction with the reservoir generated a renormalization of the mass but also a *retarded interaction* in the effective action controled by the kernel

$$K(\tau) = \frac{2}{\pi\hbar\beta} \sum_{n=-\infty}^{\infty} \int_0^{\infty} d\omega\, \frac{I(\omega)}{\omega}\, \frac{\nu_n^2}{\nu_n^2+\omega^2}\, \exp(i\nu_n\tau) , \tag{13}$$

with $I(\omega)$ the spectral density of the bath, see Eq. (33) below for its definition, and ν_n the Matsubara frequencies, $\nu_n = 2\pi n/\hbar\beta$, and n an integer in $[-\infty, \infty]$. The counterterm cancels the mass renormalization but the last retarded interaction in (12) remains.

The effect of the quantum bath is more cumbersome than in the classical case and it can lead to rather spectacular effects. A well-known example is the localization transition, as function of the coupling strength to an Ohmic bath (see its definition in Section 2.3) in the behaviour of a quantum particle in a double well potential [26]. In the context of glassy systems, the locus of the transition between the ordered and the disordered phase depends strongly on the coupling to the bath and on the type of bath considered [19], as discussed in Setion 4.

2.2 Dynamics

We shall see that the distinction between the effect of a reservoir on the statistical properties of a classical and quantum system is absent from a fully dynamic treatment. In both classical and quantum problems, the coupling to an environment leads to a *retarded* interaction. In classical problems one generally argues that the retarded interaction can be simply replaced by a local one, *i.e* one uses white baths, but in the quantum problems one cannot do the same.

2.2.1 Classical problem

The dissipative dynamics of a classical system in contact with an environment is usually described by a phenomenological Langevin equation. If the system is simply given by a particle of mass M, whose position is denoted by q, the Langevin equation reads

$$\begin{aligned} M\ddot{q}(t) + \int_0^t dt' \gamma(t-t')\dot{q}(t') &= -V'(q(t)) + \xi(t) & (14) \\ \langle \xi(t)\xi(t') \rangle &= 2k_B T \gamma(t-t') & (15) \end{aligned}$$

where $\gamma(t-t')$ is a retarded friction, $\xi(t)$ is a time-dependent Gaussian random force with zero mean and correlation given by Eq. (15).

This equation can be derived from the coupled system defined in Eq. (1) [27, 16]. Indeed, if one assumes that the initial coordinates and momenta of the oscillators have a canonical distribution at an inverse temperature β (shifted by the coupling to the particle $x_a(0) \to x_a(0) - c_a/(m_a x_a^2) q(0)$), Newton's equation for the evolution of the particle becomes Eq. (14). The random nature of the force ξ is due to the randomicity in the initial values of the position and momenta of the oscillators. The use of an equilibrium measure for the distribution of the oscillators implies the invariance under time translations of the friction kernel, $\gamma(t-t')$, and its relation to the noise-noise correlation in Eqs. (14) and (15). The latter is a fluctuation-dissipation theorem that holds for the bath variables. Different forms of $\gamma(t-t')$ can be generated by different choices of the ensemble of oscillators in the bath. A white bath corresponds to $\gamma(t-t') = \gamma\delta(t-t')$ but coloured cases are also possible.

Once the Langevin equation has been established on a firmer ground, it can be used as a starting point to study the dynamics of more complicated classical systems in contact with classical thermal baths. The dynamics of a macroscopic system in contact with an environment is then given by the number of dynamic degrees of freedom, N, coupled Langevin equations. Diagrammatic techniques, the use of a dynamic generating functional, etc. yield the dynamic equations for macroscopic correlation and response functions.

The dynamic generating functional is defined as [28]

$$Z_{\rm DYN}[\eta] \equiv \int \mathcal{D}\xi(t) \exp\left[-\frac{1}{4k_bT}\int_0^t dt' \int_0^t dt''\, \xi(t')\gamma^{-1}(t'-t'')\xi(t'') + \int_0^t dt'\, \eta(t') q_\xi(t')\right]$$

with $q_\xi(t)$ the solution to the Langevin equation. We have set the initial time to be $t_0 = 0$ and we follow the evolution up to a generic time t. A very useful expression for it, usually called the Martin-Siggia-Rose generating functional, is obtained by introducing an identity

$$1 = \int \mathcal{D}q(t) \;\; \delta\left(M\ddot{q}(t) + \int_0^t dt' \gamma(t-t')\dot{q}(t') + V'(q(t)) - \xi(t)\right) \; |\text{det}| \; . \tag{16}$$

The factor $|\text{det}|$ is the determinant of the operator $\delta(t-t')(M\partial_t^2 + V''(q)) + \gamma(t-t')\partial_{t'}$ and ensures that the integral equals one. The delta function can be exponentiated with an auxiliary field. The determinant can be exponentiated with time-dependent anticommunting variables but, since it does not yield a relevant contribution to the dynamic equations we forget it. $Z_{\rm DYN}$ reads

$$\begin{aligned}
Z_{\rm DYN}[\eta(t), \hat\eta(t)] \equiv & \int \mathcal{D}\xi(t)\mathcal{D}q(t)\mathcal{D}i\hat{q}(t) \\
& \exp\left[\int_0^t dt'\, i\hat{q}(t') \left(M\ddot{q}(t') + \int_0^t dt'' \gamma(t'-t'')\dot{q}(t'') + V'(q(t')) - \xi(t')\right)\right] \\
& \times \exp\left[-\frac{1}{4k_bT}\int_0^t dt' \int_0^t dt''\, \xi(t')\gamma^{-1}(t'-t'')\xi(t'') \right. \\
& \qquad \left. + \int_0^t dt' \; (\eta(t')q(t') + \hat\eta(t') i\hat{q}(t'))\right]
\end{aligned}$$

where we have introduced a new source $\hat\eta(t)$, coupled to the auxiliary field $i\hat{q}(t)$. The integration over the noise $\xi(t)$ is Gaussian and it can be readily done, it yields an interaction between the auxiliary fields

$$k_B T \int_0^t dt' \int_0^t dt''\; i\hat{q}(t')\, \gamma(t'-t'')\, i\hat{q}(t'') \; . \tag{17}$$

In the usual white noise case, this term reduces to the local expression $k_B T \int dt' (i\hat{q}(t'))^2$. If, more generally, the bath is coloured, the interaction with the environment generates a *retarded* interaction in the effective action.

The macroscopic correlations and linear responses are given by

$$C(t,t') = \langle q(t) q(t')\rangle_\xi = \frac{1}{Z_{\rm DYN}[\eta,\hat\eta]} \left.\frac{\delta^2 Z_{\rm DYN}[\eta,\hat\eta]}{\delta\eta(t)\delta\eta(t')}\right|_{\eta=\hat\eta=0} \tag{18}$$

$$R(t,t') = \left.\frac{\delta\langle q(t)\rangle_\xi}{\delta h(t')}\right|_{h=0} = \langle i\hat{q}(t) q(t')\rangle_\xi = \frac{1}{Z_{\rm DYN}[\eta,\hat\eta]} \left.\frac{\delta^2 Z_{\rm DYN}[\eta,\hat\eta]}{\delta\eta(t)\delta\hat\eta(t')}\right|_{\eta=\hat\eta=0} \tag{19}$$

with $h(t')$ a small field applied at time t' that modifies the potential energy $V \to V - h(t')q(t')$. If the system has quenched random exchanges or any kind of disorder, one may be interested in calculating the averaged correlations and responses over different realizations of disorder. Surprisingly, this average is very easy to perform in a dynamic calculation [29]. The normalization factors $1/Z_{\rm DYN}[\eta,\hat\eta]$ in (18) and (19) have to be evaluated at zero external sources where they are trivially independent of the random interactions. Hence, it is sufficient to average $Z_{\rm DYN}[\eta,\hat\eta]$ over disorder and then take variations with respect to the sources to derive the thermal and disorder averaged two-point functions. This property contrasts with an equilibrium calculation where the expectation values are given by $[\langle A\rangle] = [1/Z \sum_{\rm CONF} A \exp(-\beta H)]$, with $[\cdot]$ denoting the disorder average. In this case, the partition function Z depends explicitly on the random exchanges and one has to introduce replicas [2, 30] to do the averaging. [31]

Having assumed the initial equilibration of the environment ensures that a normal system will eventually equilibrate with it. The interaction with the bath allows the system to dissipate its energy and to relax until thermalization is reached. However, in some interesting cases, as the glassy models discussed in [1] and below, the time needed to equilibrate is a fast growing function of N, the number of dynamic degrees of freedom. Thus, the evolution of the system in the thermodynamic limit occurs out of equilibrium. In real systems, a similar situation occurs when the equilibration time crosses over the observation time and falls out of the experimental time-window.

A final interesting remark about the relevance of quenched disorder is the following. When a system with quenched disorder evolves out of equilibrium at finite temperature, the correlation function and the response function do not depend on the realization of disorder if the size of the system is large enough (the realization of disorder has to be a typical one). These quantities are self-averaging. This statement is easily checked in a simulation. When times get sufficiently long as to start seeing the approach to equilibrium, dependences on the realizations of disorder appear.

2.3 Quantum fluctuations

The Schwinger-Keldysh formalism [**?**] allows one to analyse the real-time dynamics of a quantum system. The starting point is the time dependent density operator

$$\hat\rho_{\rm TOT}(t) = \exp\left(-\frac{i}{\hbar}\hat H_{\rm TOT} t\right)\; \hat\rho_{\rm TOT}(0)\; \exp\left(\frac{i}{\hbar}\hat H_{\rm TOT} t\right) \; . \qquad (20)$$

We have set the initial time t_o to be $t_o = 0$. Introducing identities, an element of the time-dependent density matrix reads

$$\begin{aligned}
&\rho(q'', x''_a; q', {x'}_a; t) = \\
&\int_{-\infty}^{\infty} dQ dQ' dX_a dX'_a \; \langle q'', x''_a| \; \exp\left(-\frac{i}{\hbar}\hat H_{\rm TOT} t\right) \; |Q, X_a\rangle \; \langle Q, X_a| \; \hat\rho_{\rm TOT}(0) \; |Q', X'_a\rangle \\
&\qquad \times \langle Q', X'_a| \; \exp\left(\frac{i}{\hbar}\hat H_{\rm TOT} t\right) \; |q', x'_a\rangle \; . \qquad (21)
\end{aligned}$$

The first factor is the coordinate representation of the evolution operator $\exp(-i\hat{H}t/\hbar$ and it can be represented as functional integral. The third term can also be represented in functional form. They read:

$$\langle q'', x''_a|\,\exp\left(-\frac{i}{\hbar}\hat{H}_{\rm TOT}t\right)\,|Q, X_a\rangle \;=\; \int_Q^{q''}\mathcal{D}q^+(t)\int_{X_a}^{x''_a}\mathcal{D}x_a^+(t)\;\exp\left(\frac{i}{\hbar}S^+_{\rm TOT}\right) \quad (22)$$

$$\langle Q', X'_a|\,\exp\left(\frac{i}{\hbar}\hat{H}_{\rm TOT}t\right)\,|q', x'_a\rangle \;=\; \int\mathcal{D}q^-(t)\int\mathcal{D}x_a^-(t)\;\exp\left(-\frac{i}{\hbar}S^-_{\rm TOT}\right)\;. \quad (23)$$

Interestingly enough, the evolution operator in Eq. (23) gives rise to a path integral going backwards in time, from $q^-(t) = q', x_a^-(t) = x'_a$ to $q^-(0) = Q', x_a^-(0) = X'_a$. The full time-integration can then be interpreted as being closed, going forwards from $t_0 = 0$ to t and then backwards from t to $t_0 = 0$. This motivates the name "closed time path formalism". A doubling of degrees of freedom (q^+, q^-) appeared and it is intimately linked to the introduction of Lagrange multipliers in the functional representation of the Langevin dynamics in the classical limit.

The action $S_{\rm TOT}$ has the usual four contributions, from the system, the reservoir, the interaction and the counterterm. The action of the system reads

$$S^+_{\rm SYST}[\eta^+(t)] = \int_0^t dt'\,\left[\frac{M}{2}\left(\frac{q^+(t')}{dt'}\right)^2 - V(q^+(t')) + \eta^+(t')q^+(t')\right] \quad (24)$$

where we have introduced a time-dependent source $\eta^+(t)$. Similarly, we introduce a source $\eta^-(t)$ in the path integral going backwards in time.

As usual we are interested in the dynamics of the system under the effect of the reservoir. Hence, we compute the reduced density matrix

$$\rho_{\rm RED}(q'', q'; t) = \int_{-\infty}^{\infty} dx_a\,\langle q'', x_a|\,\hat{\rho}_{\rm TOT}(t)\,|q', x_a\rangle\;. \quad (25)$$

The initial density operator $\hat{\rho}(0)$ has the information about the initial state of the whole system. If one assumes that the system and the bath are set in contact at the initial time, the operator factorizes

$$\hat{\rho}(0) = \hat{\rho}_{\rm SYST}(0)\hat{\rho}_{\rm ENV}(0)\;. \quad (26)$$

(Other initial preparations, where the factorization does not hold, can also be considered and may be more realistic in certain cases.) If the environment is initially in equilibrium at an inverse temperature β,

$$\hat{\rho}_{\rm ENV}(0) = \frac{1}{Z_{\rm ENV}}\exp(-\beta\hat{H}_{\rm ENV})\;, \quad (27)$$

the dependence on the bath variables is quadratic; they can traced away to yield the reduced density matrix:

$$\begin{aligned}\rho_{\rm RED}(q''; q'; t) \;=\; & \int_{-\infty}^{\infty} dQ\int_{-\infty}^{\infty} dQ'\int_{q^+(0)=Q}^{q^+(t)=q''}\mathcal{D}q^+(t)\int_{q^-(0)=Q'}^{q^-(t)=q'}\mathcal{D}q^-(t) \\ & \exp\left(\frac{i}{\hbar}S_{\rm EFF}\right)\langle Q|\,\hat{\rho}_{\rm SYST}(0)\,|Q'\rangle \end{aligned} \quad (28)$$

with the effective action

$$S_{\text{EFF}} = S^{+}_{\text{SYST}} - S^{-}_{\text{SYST}} + S_{\text{TH}} \ . \tag{29}$$

The last term has been generated by the interaction with the environment and it reads

$$\begin{aligned} S_{\text{TH}} &= -\int_0^t dt' \int_0^t dt'' \, \left(q^+(t') - q^-(t')\right) \, \eta(t'-t'') \, \left(q^+(t'') + q^-(t'')\right) \\ &\quad + i \int_0^t dt' \int_0^{t'} dt'' \, \left(q^+(t') - q^-(t')\right) \, \nu(t'-t'') \, \left(q^+(t'') - q^-(t'')\right) \ . \end{aligned} \tag{30}$$

The noise and dissipative kernels ν and η are given by

$$\nu(t) = \int_0^\infty d\omega I(\omega) \coth\left(\frac{1}{2}\beta\hbar\omega\right) \cos(\omega(t)) \ , \tag{31}$$

$$\eta(t) = \theta(t)\frac{d\gamma(t)}{dt} = -\theta(t) \int_0^\infty d\omega \, I(\omega) \sin(\omega(t)) \ . \tag{32}$$

In these equations, $I(\omega)$ is the spectral density of the bath:

$$I(\omega) = \frac{\pi}{2} \sum_{a=1}^{N_b} \frac{c_a^2}{m_a \omega_a} \delta(\omega - \omega_a) \ , \tag{33}$$

that is a smooth function of ω usually taken to be

$$I(\omega) = \gamma\omega \left(\frac{\omega}{\omega_s}\right)^{s-1} \exp(-\omega/\Lambda) \tag{34}$$

with γ the “friction coefficient”, ω_s a constant, Λ a high frequency cut-off and s a parameter that characterises different baths: $s = 1$ is Ohmic (and leads to the usual white noise when $\Lambda \to \infty$), $s > 1$ is super-Ohmic and $s < 1$ is sub-Ohmic. As usual, a counterterm cancels the mass renormalization.

Next, we have to choose an initial density matrix for the system. One natural choice, having in mind the quenching experiments usually performed in classical system, is the diagonal matrix

$$\langle Q| \, \hat{\rho}_{\text{SYST}}(0) \, |Q'\rangle = \delta(Q - Q') \tag{35}$$

that corresponds to a random “high-temperature” situation and that simplifies considerably the expression in (28).

In the classical limit, $\hbar \to 0$, the effective action reduces to the one in the functional representation of a Langevin process with coloured noise. This can be easily verified by renaming the variables

$$q \equiv \frac{1}{2}(q^+ + q^-) \qquad i\hat{q} \equiv \frac{1}{\hbar}(q^+ - q^-) \ , \tag{36}$$

expanding in powers of $\hbar$, and keeping only the leading terms. If, moreover, the limit $\Lambda \to \infty$ is taken, the kernels γ and ν become proportional to δ functions and one recovers a white noise.

Any two point correlation functions that depends only on the system is then given by

$$\langle \hat{A}(t)\hat{B}(t')\rangle = \frac{\mathrm{Tr}\hat{A}(t)\hat{B}(t')\hat{\rho}_{\mathrm{RED}}(0)}{\mathrm{Tr}\hat{\rho}_{\mathrm{RED}}(0)} \; . \tag{37}$$

As in the classical stochastic problem, the normalization in given by a quantity that is trivially independent of disorder, $\mathrm{Tr}\hat{\rho}_{\mathrm{RED}}(0) = 1$, since we have chosen $\hat{\rho} = 1$. Hence, there is no need to introduce replicas in such a quantum dynamic calculation (if the initial conditions are not correlated with disorder).

Clearly $\langle \hat{A}(t)\hat{B}(t') \neq \hat{B}(t')\hat{A}(t)\rangle$ and one can define symmetrized and anti-symmetrized correlations:

$$C_{\{A,B\}}(t,t') = \frac{1}{2}\langle A(t)B(t') + B(t')A(t)\rangle \; , \tag{38}$$

$$C_{[A,B]}(t,t') = \frac{1}{2}\langle A(t)B(t') - B(t')A(t)\rangle \; , \tag{39}$$

respectively.

If one generalizes the above system to be one described by a field ϕ_i with $i = 1, N$ components, the symmetrized correlation function is given by

$$C_{ij}(t,t') \equiv \frac{1}{2}\langle \phi_i(t)\phi_j(t') + \phi_j(t')\phi_i(t)\rangle \tag{40}$$

$$= \frac{\hbar^2}{2}\left[\frac{\delta^2}{\delta\eta_i^+(t)\delta\eta_j^-(t')} + \frac{\delta^2}{\delta\eta_j^+(t')\delta\eta_i^-(t)}\right]\rho_{\mathrm{RED}}\Big|_{\eta=0} \tag{41}$$

($C_{ij}(t,t') = C_{ji}(t',t)$). The linear response function $R_{ij}(t,t')$ is defined as the variation of the averaged field $\langle\phi_i\rangle$ with respect to the "strength" f_j of a perturbation that modifies the potential energy according to $V \to V - \sum_{i=1}^N f_i\phi_i$:

$$R_{ij}(t,t') \equiv \left.\frac{\delta\langle\phi_i(t)\rangle}{\delta f_j(t')}\right|_{f=0} \; . \tag{42}$$

In linear-response theory, it can be expressed in terms of the averaged commutator:

$$R_{ij}(t,t') = \frac{i}{\hbar}\theta(t-t')\langle[\hat{\phi}_i(t),\hat{\phi}_j(t')]\rangle \tag{43}$$

$$= \frac{\hbar}{i}\left[\frac{\delta^2}{\delta\eta_i^+(t)\delta\eta_j^+(t')} + \frac{\delta^2}{\delta\eta_j^+(t)\delta\eta_i^-(t')}\right]\rho_{\mathrm{RED}}\Big|_{\eta=0} \; . \tag{44}$$

It is also useful to write the correlation and response in terms of the fields ϕ_i^+, ϕ_i^-:

$$C_{ij}(t,t') = \frac{1}{2}\langle \phi_i^+(t)\phi_j^-(t') + \phi_j^+(t')\phi_i^-(t)\rangle \; , \tag{45}$$

$$R_{ij}(t,t') = \frac{i}{\hbar}\langle \phi_i^+(t)\left(\phi_j^+(t') - \phi_j^-(t')\right)\rangle \; . \tag{46}$$

3 The quantum fluctuation-dissipation theorem

Proofs and descriptions of the quantum FDT can be found in several textbooks [16, 33]. We recall its expression in the time-domain and in a mixed time-Fourier notation that gave us insight as to how extend it to the case of glassy non-equilibrium dynamics [15].

If at time t' the system is characterized by a density functional $\rho(t')$, the two-time correlation functions are given by Eq. (37), (38) and (39). In linear response theory $R_{AB}(t,t')$ and the correlation $C_{[A,B]}(t,t')$ are related by the Kubo formula

$$R_{AB}(t,t') = \frac{i}{\hbar}\,\theta(t-t')\,\langle\,[A(t),B(t')]\,\rangle = \frac{2i}{\hbar}\,\theta(t-t')\,C_{[A,B]}(t,t')\;. \tag{47}$$

If the system has reached equilibrium with a heat-bath at temperature T at time t', the density functional $\rho(t')$ is just the Boltzmann factor $\exp(-\beta H)/Z$. It is then immediate to show that, in equilibrium, time-translation invariance and and the KMS properties hold:

$$C_{AB}(t,t') = C_{AB}(t-t') \tag{48}$$

$$C_{AB}(t,t') = C_{BA}(t',t+i\beta\hbar) = C_{BA}(-t-i\beta\hbar,-t')\;. \tag{49}$$

Using now the KMS properties and assuming, for definiteness, that $t>0$ it is easy to verify the following equation

$$C_{\{A,B\}}(\tau) + \frac{i\hbar}{2}R_{AB}(\tau) = C_{\{A,B\}}(\tau^*) - \frac{i\hbar}{2}R_{AB}(\tau^*)\;, \tag{50}$$

where $\tau = t+i\beta\hbar/2$. This is a way to express FDT through an analytic continuation to complex times.

In terms of the Fourier transformed $C_{AB}(\omega)$ defined by

$$C_{AB}(t-t') = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\exp(-i\omega(t-t'))\,C_{AB}(\omega) \tag{51}$$

the KMS relations read

$$C_{AB}(\omega) = \exp(\beta\hbar\omega)C_{BA}(-\omega) \tag{52}$$

and lead to the following relation between Fourier transforms of the correlation functions:

$$\begin{aligned} 2C_{[A,B]}(\omega) &= (1-\exp(-\beta\hbar\omega))\;C_{AB}(\omega)\;,\\ 2C_{\{A,B\}}(\omega) &= (1+\exp(-\beta\hbar\omega))\;C_{AB}(\omega)\;,\\ C_{[A,B]}(\omega) &= \tanh\left(\frac{\beta\hbar\omega}{2}\right)\,C_{\{A,B\}}(\omega)\;. \end{aligned} \tag{53}$$

Back in Eq. (47) this implies FDT

$$R_{AB}(t-t') = \frac{i}{\hbar}\,\theta(t-t')\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\;\exp(-i\omega(t-t'))\;\tanh\left(\frac{\beta\hbar\omega}{2}\right)\;C_{\{A,B\}}(\omega) \tag{54}$$

Using

$$\int_0^\infty dt\, \exp(i\omega t) = \lim_{\epsilon\to 0^+} \frac{i}{\omega + i\epsilon} = \pi\delta(\omega) + i\frac{P}{\omega} \tag{55}$$

one has

$$R_{AB}(\omega) \;=\; -\frac{1}{\hbar} \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} \frac{d\omega'}{\pi}\, \frac{1}{\omega - \omega' + i\epsilon}\, \tanh\left(\frac{\beta\hbar\omega'}{2}\right) C_{\{A,B\}}(\omega') \tag{56}$$

from which we obtain the real and imaginary relations

$$\begin{aligned} \mathrm{Im} R_{AB}(\omega) &= \frac{1}{\hbar} \tanh\left(\frac{\beta\hbar\omega}{2}\right) C_{\{A,B\}}(\omega')\ , \\ \mathrm{Re} R_{AB}(\omega) &= -\frac{1}{\hbar}\, P \int_{-\infty}^{\infty} \frac{d\omega'}{\pi}\, \frac{1}{\omega - \omega'}\, \tanh\left(\frac{\beta\hbar\omega'}{2}\right) C_{\{A,B\}}(\omega')\ . \end{aligned} \tag{57}$$

If $\beta\hbar\omega/2 \ll 1$, $\tanh(\beta\hbar\omega/2) \sim \beta\hbar\omega/2$ and Eq. (54) becomes the classical FDT:

$$R_{AB}(\tau) = -\frac{1}{T}\frac{dC_{AB}(\tau)}{d\tau} \tag{58}$$

with $\tau = t - t'$.

We shall see in the next section how this theorem is modified in a quantum glassy model.

4 A toy model

A standard toy model for glassy system is the p-spin disordered model. It realizes many of the expected features of classical glasses [1]. We have studied [15, 17]-[19] a quantum extension of it in which we reinterpreted the continuous spins, q_i, as coordinate operators and introduce canonically conjugate momentum operators, π_i. Coordinate and momentum operators verify the usual commutation relations

$$[q_i, q_j] = [p_i, p_j] = 0\ \ , \qquad [p_i, q_j] = -i\hbar\delta_{ij}\ . \tag{59}$$

As shown in [17] this model is also a soft spin limit of the mean-field quantum Ising model with p spin interations in a transverse field. The Hamiltonian of the system, H_{SYST}, is defined by adding to the usual potential energy a "kinetic energy" term and reads

$$H_{\text{SYST}} = \frac{p^2}{2M} - \sum_{i_1<\ldots<i_p}^{N} J_{i_1\ldots i_p} q_{i_1}\ldots q_{i_p}\ . \tag{60}$$

We denoted $p^2 = \vec{p}\cdot\vec{p}$, $q^2 = \vec{q}\cdot\vec{q}$ with $\vec{p} = (p_1, \ldots, p_N)$ and $\vec{q} = (q_1, \ldots, q_N)$. The mass M can be linked to the inverse of the transverse field in the soft spin limit of the Ising model. A Lagrange multiplier z enforces the mean spherical constraint,

$$\frac{1}{N}\sum_{i=1}^{N}\langle q_i^2\rangle = 1, \tag{61}$$

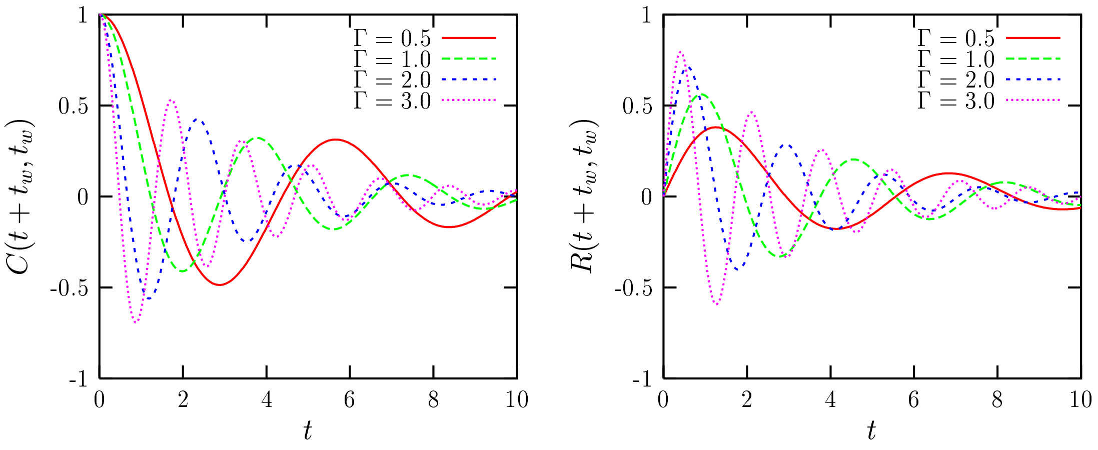

Figure 1: The (stationary) symmetrized correlation and linear response in the disordered phase for several values of the quantum parameter Γ. From Ref. [19].

where the angular brackets denote the thermodynamic average. The interaction strengths $J_{i_1 \dots i_p}$ are taken from a Gaussian distribution with zero mean and variance $\overline{\left(J_{i_1 \dots i_p}\right)^2} = \tilde{J}^2 p!/(2N^{p-1})$; henceforth the overline represents the average over disorder. Consequentely, the second term in H_{SYST} is a random Gaussian potential energy with zero average and correlation

$$\overline{V(\vec{q})V(\vec{q}')} = \frac{\tilde{J}^2 N}{2} \left(\frac{\vec{q} \cdot \vec{q}'}{N} \right)^p \; . \tag{62}$$

We used an oscillator bath as the one in Eq. (2) with a linear interaction between system and bath.

The behaviour of the quantum system is very rich. A short summary of results is the following:

- The model has an ordered and a disordered phase separated by a critical surface in the $(T, \Gamma \equiv \hbar^2/(M\tilde{J}), \alpha)$ parameter space, where T is temperature, Γ measures the strength of the quantum fluctuations in the system and α is the strength of the coupling to the bath. The location of the transition also depends on the parameter s that distinguishes different types of reservoirs (Ohmic, super-Ohmic, sub-Ohmic).

- Actually, there are two transition surfaces, a dynamic and a static one. The former is obtained from the real-time dynamic analysis while the latter follows from the study of the partition function using the Matsubara formalism. (Actually, partial information about the nonequilibrium dynamics of the model can also be obtained from the replicated Matsubara formalism if a special *Ansatz* for the replica matrices, that does not correspond to equilibrium, is used.) Below the dynamic transition the system is out of equilibrium. In the classical limit this double transition was already known. It survives when quantum fluctuations are present.

- The nature of the dynamic and static transitions changes when the temperature decreases: close to the classical critical point ($\Gamma \to 0$), the transitions are discontinuous but of second order thermodynamically. There are no discontinuities in the thermodynamic quantities but there is a plateau that develops in the correlation function where the transition is approached from above. This is the expected behaviour in classical glasses. Spin-glasses have continuous transitions (without precursors). Conversely, close to the quantum critical point ($T = 0, \Gamma$ finite, α fixed) the transitions become of first order thermodynamically, *e.g.* the energy density jumps. A change in the order of the transition in a quantum glass has been reported experimentally by Aeppli, Rosenbaum and collaborators.

- In the classical case, the study of the Thouless-Anderson-Palmer (TAP) free energy landscape has been of mcuh help to understand the behaviour of these systems. A TAP approach can also be developed for the quantum problem [18]. It helps understanding the change in nature of the transition close to the quantum critical point.

- In the disordered phase the dynamics is fast and it occurs in equilibrium. The auto correlation and the linear response functions are invariant under time-translations. They both show oscillations, as is typical of a quantum problem. The frequency of the oscillations depends on Γ and the characteristics of the bath. Correlations and responses are linked by the quantum fluctuation dissipation theorem FDT. At high temperatures and after a short transient there is a decoherence effect and the dynamics becomes totally classical. For example, responses and correlations are related by the classical FDT.

- In the ordered phase the dynamics is very slow and the system is not equilibrated with its environment. The autocorrelation functions decay in two steps, with a fast decay towards a plateau and a slow decay below the plateau towards zero, as a function of the time difference t and for fixed waiting-time. The first decay is stationary while in the second decay the waiting-time dependences exist for all times if the thermodynamic limit has been taken at the outset. A few quantum oscillations appear in the first step of the relaxation and they later disappear. It is important to note that these oscillations, as functions of the time difference, will be present for any arbitrarily large value of the waiting time.

- The comparison of responses and correlations in the ordered phase is particularly interesting. The quantum FDT is a complicated integral relation between the symmetrized correlation and the linear response. This relation holds for time-differences that are short as compared with the waiting time. These correspond to the range of time-differences when the correlation decays from 1 to

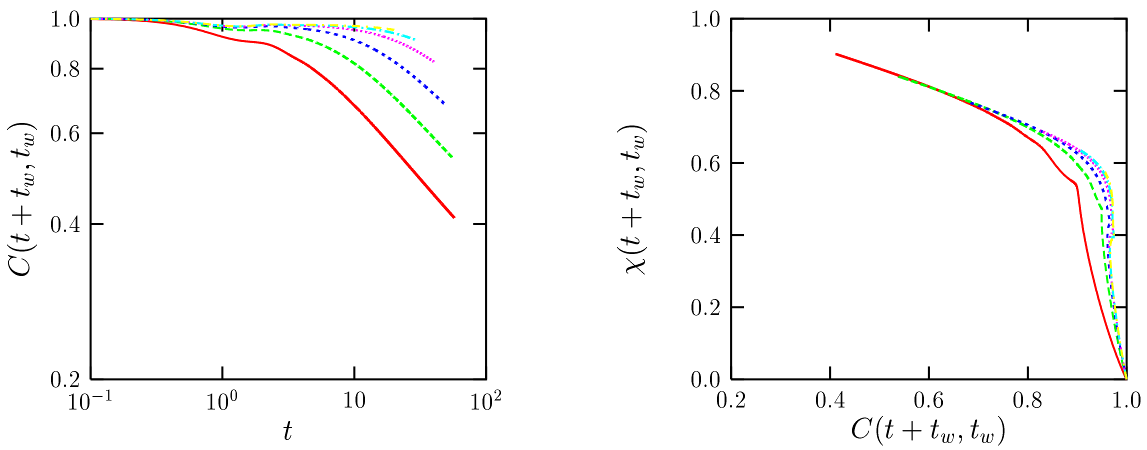

Figure 2: The aging symmetrized correlation (left). The comparison between integrated response and symmetrized correlation (right). Different curves correspond to different values of the waiting time. For the slow part of the relaxation an effective temperature generated as shown by the finite slope in the parametric curves. From Ref. [15].

the plateau. In the second decay the quantum FDT is no longer verified, much as it happens in the classical problem. A comparison of the integrated responses and the symmetrized correlation in the parametric manner explained in [1] yields a surprising result at face value: the two quantities are related by a *classical* FDT with an effective temperature T_{EFF} that depends on the parameters in the problem T, Γ, α. A non zero effective temperature appears even when the temperature of the bath is zero (but the coupling to the environment still exists, $\alpha > 0$).

- The generation of a non-trivial effective temperature for the slow part of the decay gives support to the "decoherent" effect already observed in the decay of the correlations. If the second decay is controlled by a nontrivial, rather high, effective temperature, it is natural that it will look classical in all respects.

- The similarity between the second decay in the classical and quantum problem can be argued as follows (it can also be shown explicitly by solving the dynamic equations). The response functions decay rather fast to zero when the time difference increases (the response functions integrated over a time-interval of very long length do not vanish though, this is the "weak long-term memory" property). The explicit dependence on $\hbar$ in the regime of widely separated times comes from factors with higher powers of R that vanish. The effect of quantum fluctuations on the slow time-difference regime is simply to renormalize [34] certain parameters in the equations that look classical when the asymptotic limit is taken.

A similar dynamic behaviour in the glassy phase has been obtained by Chamon, Kenneth and Yue for the soft version of the quantum Sherrington-Kirkpatrick

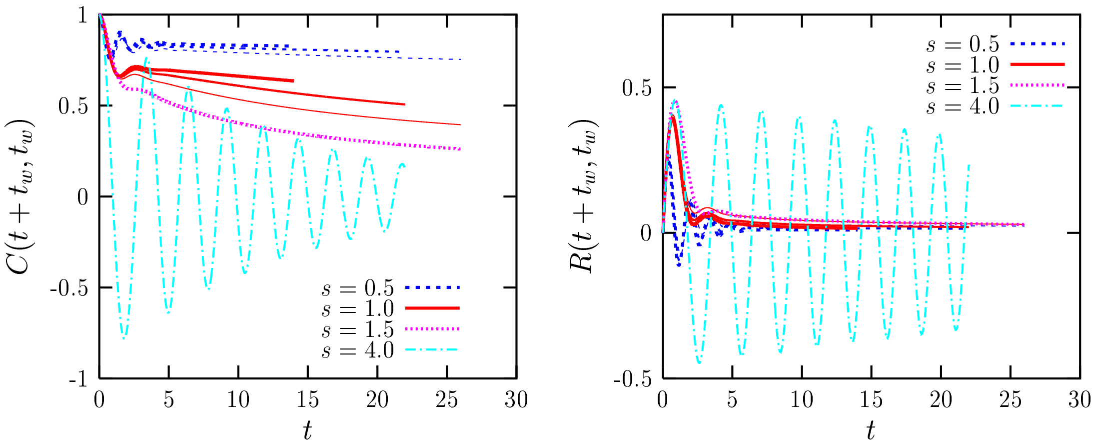

Figure 3: The dependence of the decay on the type of bath used. Different curves correspond to different values of s. From Ref. [19].

model [34] and by Biroli and Parcollet for the fully connected SU(N) Heisenberg model in the large N limit [35].

The analytical solution [36] to the nonequilibrium dynamics of simple classical glassy models (as the classical limit of the toy model considered here) has lead to the definition of the effective temperature [1, 13] that was later found in more realistic models of glasses (finite dimensional spin-glass model, Lennard-Jones mixtures, etc.) and that is now starting to be explored experimentally. Even if the models treated analytically are rather simple and have many defects as being mean-field in nature they have lead one to develop new ideas. The fact that very different systems when they evolve out of equilibrium by in a slow manner have rather similar dynamics is justified by the self-generation of effective temperatures that control their slow nonequilibrium relaxation. Thus, one can see a similar decay in classical glasses, quantum glasses, gently vibrated granular matter or sheared liquids.

For the quantum problem, it is now a challenge to check if similar results to the ones described here hold in finite dimensional glassy models. The lack of performant computer simulations techniques to follow the real time dynamics of macroscopic quantum systems renders the develoment of analytical approaches even more necessary and interesting than in the classical limit.

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[22] Note that the sign of the mass term generated by the coupling to the environment is negative. If strong enough, it may render unstable an initially stable potential well. This is a very important effect. One way to avoid this problem is to introduce the counterterm. One could also think that the coupling to the bath must be very weak and that the last term must vanish by itself (say, all $c_a \to 0$). However, this might be problematic when dealing with the stochastic dynamics since a very weak coupling to the bath implies also a very slow relaxation. It is then conventional to include the counterterm to cancel the mass renormalization.

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